Fixed-point theorems for topological spaces, simplicial complexes and graphs

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- Sign patterns and the Brouwer fixed point theorem
- Lefschetz fixed point theorem:
  - spaces
  - simplicial complexes
  - graphs
- time permitting: Lefschetz-Hopf fixed point formula

Theorem (L. E. J. Brouwer, 1910)

Let B_n be the n-dimensional unit ball $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leqslant 1\}$.
Let $f : B_n \to B_n$ be any continuous function.
Then $f(x) = x$ for some point $x \in B_n$.

Remark If a topological space X has the above *fixed-point property* then so does any space homeomorphic to X.

In particular, the Brouwer fixed point theorem is valid for any compact, convex subset of $\mathbb{R}^n$.

Brouwer’s sensory intuitions

The theorem is supposed to have originated from Brouwer’s observation of a cup of coffee. If one stirs to dissolve a lump of sugar, it appears there is always a point without motion. He drew the conclusion that at any moment, there is a point on the surface that is not moving.

Brouwer is said to have added: “I can formulate this splendid result different, I take a horizontal sheet, and another identical one which I crumple, flatten and place on the other. Then a point of the crumpled sheet is in the same place as on the other sheet.”

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(this is probably just me) Now matter how hard I try, I've never been *less* convinced in my life than by the above formulations.

one-dimensional case is easy

Let $f : [0, 1] \to [0, 1]$ be continuous.
Consider $g(x) = f(x) - x$.
$g(0) \geqslant 0$ and $g(1) \leqslant 0$.

By the Intermediate Value Theorem, $g(z) = 0$ for some $z \in [0, 1]$. Thence $f(z) = z$.

two dimensions: heuristic

We want to prove: any continuous map $\langle f_1(x,y), f_2(x,y)\rangle : [0,1]^2 \to [0,1]^2$ has a fixed point.

Exhibiting a *total* lack of imagination, we'll do the same as in the one-dimensional case:

Let $g_1(x,y) = f_1(x,y) - x$ and $g_2(x,y) = f_2(x,y) - y$.

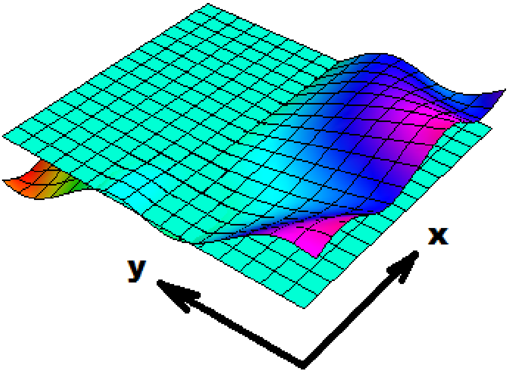
Then g_1, g_2 are continuous functions $[0,1]^2 \to \mathbb{R}$.

For all $0 \leqslant y \leqslant 1$, $g_1(0,y) \geqslant 0$ and $g_1(1,y) \leqslant 0$.
For all $0 \leqslant x \leqslant 1$, $g_2(x,0) \geqslant 0$ and $g_2(x,1) \leqslant 0$.

A point $\langle x,y\rangle$ that is a simultaneous zero of g_1 and g_2 is the same as a fixed point of $\langle f_1(x,y), f_2(x,y)\rangle$.

just intuitively

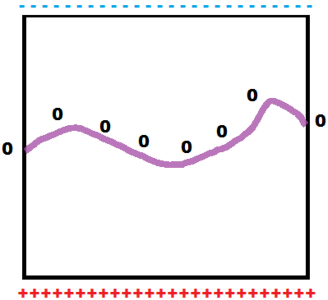
Suppose $g : [0,1]^2 \to \mathbb{R}$ is continuous such that
$g(x,0) \geqslant 0$ and $g(x,1) \leqslant 0$ for all $0 \leqslant x \leqslant 1$.



just intuitively

Suppose $g : [0,1]^2 \to \mathbb{R}$ is continuous such that
$g(x,0) \geqslant 0$ and $g(x,1) \leqslant 0$ for all $0 \leqslant x \leqslant 1$.

Such a function "ought to" have a "band of zeros" connecting the $x = 0$ and $x = 1$ edges:



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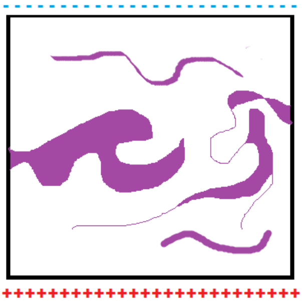
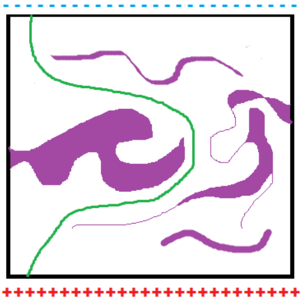
If the zero locus of g did *not* have a component connecting the $x = 0$ and $x = 1$ edges



just intuitively

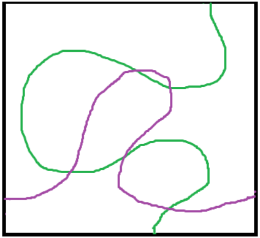
If the zero locus of g did *not* have a component connecting the $x = 0$ and $x = 1$ edges



then there would exist a zero-free path connecting a negative value of g with a positive value, contradicting the 1-dimensional case of the Intermediate Value Theorem!

just intuitively

path in the zero locus of g_1 and path in the zero locus of g_2



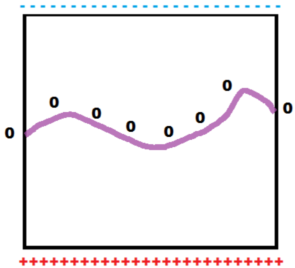
- a path connecting top to bottom and a path connecting left to right *must* intersect somewhere
- that's a common zero of g_1 and g_2
- the Brouwer fixed point theorem for $[0, 1]^2$ follows!

is our intuition correct?

Suppose $g : [0, 1]^2 \to \mathbb{R}$ is continuous such that
$g(x, 0) \geqslant 0$ and $g(x, 1) \leqslant 0$ for all $0 \leqslant x \leqslant 1$.

Let $Z = g^{-1}(0) \subseteq [0, 1]^2$.

Then there exists a continuous path $\langle p_1, p_2 \rangle : [0, 1] \to Z$ such that $p_1(0) = 0$ and $p_1(1) = 1$.



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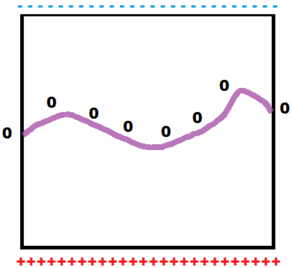

This is false.

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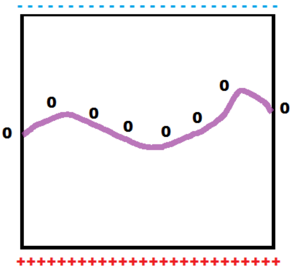

This is false.

Homework Give a counterexample.

solution to homework

Let $Z = A_1 \cup A_2 \cup A_3 \subset [0,1]^2$ where

$A_1 := \{ (x, \frac{1}{2}) \mid 0 \leqslant x \leqslant \frac{1}{2} \}$

$A_2 := \{ (\frac{1}{2}, y) \mid \frac{1}{4} \leqslant y \leqslant \frac{3}{4} \}$

$A_3 :=$ graph of $y = \frac{1}{2} + \frac{1}{4}\sin(\frac{1}{x-0.5})$ for $\frac{1}{2} < x \leqslant 1$

Z is compact so can set *dist*$\big((x,y), Z\big) := \min_{z \in Z} \text{dist}\big((x,y), z\big)$

$[0,1]^2 \setminus Z$ has two connected components, "top" (containing e.g. $(0,1)$) and "bottom" (containing e.g. $(0,0)$)

Define $g : [0,1]^2 \to \mathbb{R}$ as

$$g(x,y) := \begin{cases} -\text{dist}\big((x,y), Z\big) & \text{if } (x,y) \in \text{top component} \\ \quad 0 & \text{if } (x,y) \in Z \\ +\text{dist}\big((x,y), Z\big) & \text{if } (x,y) \in \text{bottom component} \end{cases}$$

the actual situation

Suppose $g : [0,1]^2 \to \mathbb{R}$ is continuous such that
$g(x,0) \geqslant 0$ and $g(x,1) \leqslant 0$ for all $0 \leqslant x \leqslant 1$.

Let $Z = g^{-1}(0) \subseteq [0,1]^2$.

- Then there is a connected component Z_0 of Z that intersects both $x = 0$ and $x = 1$. True.
- Then there is a path connected component Z_0 of Z that intersects both $x = 0$ and $x = 1$. False.

HAIRY!

point-set topological difficulties

- cannot always find a cross-section of the zero locus that's homeomorphic to $[0, 1]$
- topology gets yet more complicated for $[0, 1]^n$ with $n > 2$
- proof can be completed this way but needs either difficult algebraic topological machinery — more difficult than other (algebraic topological) proofs of Brouwer's fixed point theorem — or the restriction to a class of tame functions (e.g. polynomials) and compactness.
- digital topology to the rescue!

the sign pattern theorem

We'll be interested in $n \times m$ matrices, each entry of which contains two symbols:

either $+$ or $-$ (corresponding to the sign of g_1) as well as either $+$ or $-$ (corresponding to the sign of g_2)

such that

$$\text{boundary conditions} \begin{cases} \text{the first column must contain } + \\ \text{the last column must contain } - \\ \text{the bottom row must contain } + \\ \text{the top row must contain } -. \end{cases}$$

+					—
+					—
+					—
+					—
+					—

+	+	−	+	−	−
+	−	+	−	+	−
+	+	−	+	+	−
+	−	+	−	−	−
+	−	−	+	+	−

+ −	+ −	− −	+ −	− −	− −
+	−	+	−	+	−
+	+	−	+	+	−
+	−	+	−	−	−
+ +	− +	− +	+ +	+ +	− +

+ −	+ −	− −	+ −	− −	− −
+ −	− +	+ −	− −	+ +	− −
+ −	+ +	− −	+ −	+ −	− −
+ +	− +	+ −	− +	− −	− +
+ +	− +	− +	+ +	+ +	− +

Any sign matrix satisfying the boundary conditions necessarily

(a) contains a 2×2 submatrix with all four types of entries $\boxed{++}$, $\boxed{+-}$, $\boxed{-+}$ and $\boxed{--}$ (?)

(b) contains adjacent $\boxed{++}$ and $\boxed{--}$ entries (?)

(c) contains adjacent $\boxed{+-}$ and $\boxed{-+}$ entries (?)

(d) contains adjacent sign-reversed entries, that is, adjacent $\boxed{++}$ and $\boxed{--}$ *or* adjacent $\boxed{+-}$ and $\boxed{-+}$ (?)

(e) contains a 2×2 submatrix where all four symbols $+ - + -$ occur (?)

guessing a statement to prove

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true in dimension 2; probably not in higher dimensions

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(e) contains a 2×2 submatrix where all four symbols $+\ -\ +\ -$ occur

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true in dimension 2; probably not in higher dimensions

(e) contains a 2×2 submatrix where all four symbols $+ - + -$ occur
true, and so is its analogue in all dimensions.

the sign pattern theorem

Any sign matrix satisfying the boundary conditions will contain a 2×2 submatrix where all four symbols $+ - + -$ occur, e.g. ($\bullet$ = anything)

$$\begin{array}{|c|c|}\hline +- & \bullet\bullet \\ \hline -+ & \bullet\bullet \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline ++ & \bullet\bullet \\ \hline \bullet\bullet & -- \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline +\bullet & \bullet+ \\ \hline -- & \bullet\bullet \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline \bullet- & \bullet+ \\ \hline -\bullet & +\bullet \\ \hline \end{array}.$$

- can in fact guarantee that all four symbols will be found in two adjacent entries (possibly "corner adjacent")
- can prove that in dimension n, all $2n$ symbols will be found in $n + 1$ cells any two of which share a vertex
- don't know if this is optimal (but it suffices to prove the Brouwer fixed point theorem).

from discrete to continuous

Suppose g_1, g_2 are continuous functions $[0,1]^2 \to \mathbb{R}$ such that
for all $0 \leqslant y \leqslant 1$, $g_1(0, y) \geqslant 0$ and $g_1(1, y) \leqslant 0$
for all $0 \leqslant x \leqslant 1$, $g_2(x, 0) \geqslant 0$ and $g_2(x, 1) \leqslant 0$.

- for $N \in \mathbb{N}^+$, sample the domain $[0,1]^2$ at the grid points $[i/N, j/N]$, $0 \leqslant i, j \leqslant N$, and record the signs of g_1 and g_2 (0 counts as $+$ or $-$, arbitrarily)
- as $N \to \infty$, sign pattern theorem implies that for any $\epsilon > 0$ there is $\mathbf{p} \in [0,1]^2$ such that in the ϵ-neighborhood $B(\mathbf{p}, \epsilon)$ of $\mathbf{p}$, both g_1 and g_2 take on both signs
- Bolzano-Weierstrass implies that there exists $\mathbf{z} \in [0,1]^2$ such that for any $\epsilon > 0$, both g_1 and g_2 take on both signs on $B(\mathbf{z}, \epsilon)$
- continuity of g_1 and g_2 implies that $g_1(\mathbf{z}) = g_2(\mathbf{z}) = 0$
- Brouwer fixed point theorem follows
- argument generalizes to n dimensions.

approximate fixed points

Given $\epsilon > 0$, by uniform continuity there exists $\delta > 0$ such that for all $\mathbf{p}, \mathbf{q} \in [0, 1]^2$ with $\text{dist}(\mathbf{p}, \mathbf{q}) < \delta$,

$$|g_1(\mathbf{p}) - g_1(\mathbf{q})| < \epsilon \text{ and } |g_2(\mathbf{p}) - g_2(\mathbf{q})| < \epsilon\ .$$

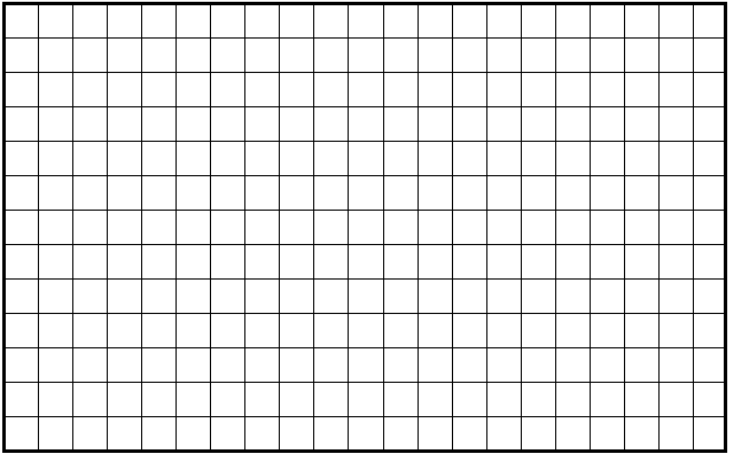
- choose $N \in \mathbb{N}$ such that $\frac{\sqrt{2}}{N} < \delta$; then, at the sample points $\mathbf{p}$ corresponding to a 2×2 sign-pattern submatrix, $|g_1(\mathbf{p})| < \epsilon$ and $|g_2(\mathbf{p})| < \epsilon$
- thus $\mathbf{p}$ is an approximate fixed point; being able to find sign patterns fast means being able to find approximate fixed points fast!
- if looking for *actual* fixed points, still need Bolzano-Weierstrass theorem or more control on the functions involved (Lipschitz conditions, smoothness etc)

proving the sign pattern theorem

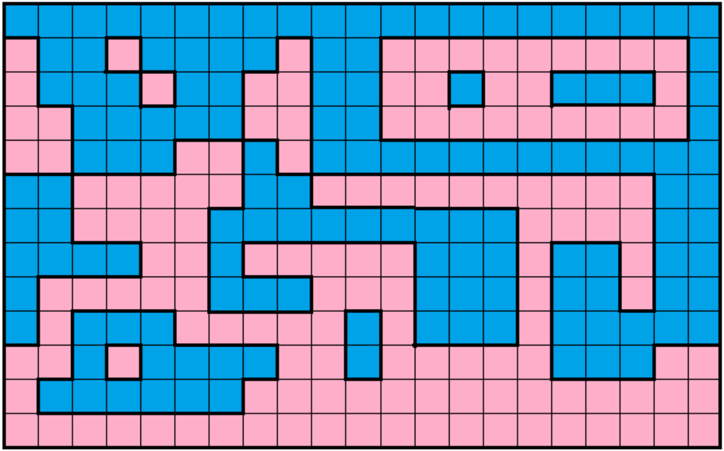
Two approaches:

- digital: direct, visual, combinatorial, hard
- algebraic: indirect “magic”, non-constructive, powerful

digital topology

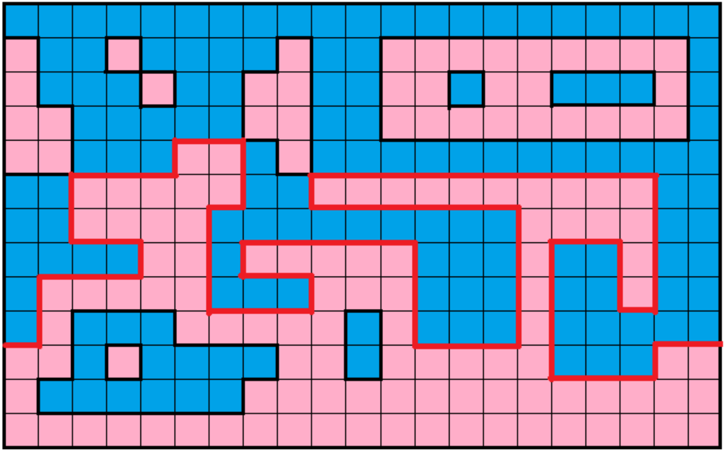


digital topology



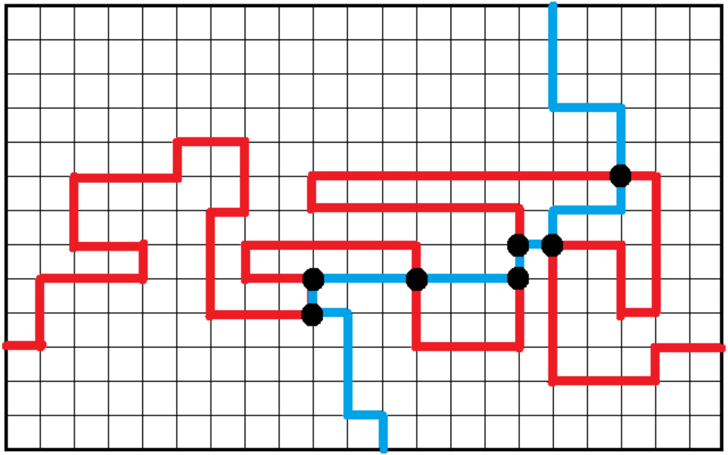
top row = blue; bottom row = pink
rest is arbitrary

digital divide



non-self-intersecting path from left to right edge
along common boundary of blue and pink cells

digital crossing

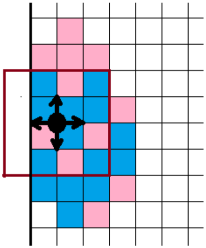


2×2 neighborhood of any vertex lying on both paths contains all four “signs” (i.e. colors)

aside on computational geometry

Naively, it seems that one can “trace out” a digital divide but I suspect that cannot be done using *local rules* alone:

Conjecture There exist no positive integer d and finite state automaton moving on the edges that senses the d-neighborhood of vertices only that has the following property: when started from the bottom left corner, it will always walk a *non-self-intersecting* path along the common boundary of “pink” and “blue” cells and stop in the bottom right corner.



one dimension . . .

Given a string of $+$ and $-$ symbols beginning with $+$ and ending with $-$

- the string contains adjacent $+$ and $-$ somewhere
- the string contains an *odd* number of sign changes
- the number of $\boxed{+ \mid -}$ substrings is one more than the number of $\boxed{- \mid +}$ substrings.

setting up the algebra

partition the four sign-pairs into three sets

$$a\left\{\;{\color{red}+}{\color{blue}+}\right.$$

$$b\left\{\;{\color{red}+}{\color{blue}-}\right.$$

$$c\left\{\;\begin{matrix}{\color{red}-}{\color{blue}+}\\ {\color{red}-}{\color{blue}-}\end{matrix}\right.$$

setting up the algebra

given sign matrix

+−	+−	−−	+−	−−	−−
+−	−+	+−	−−	++	−−
+−	++	−−	+−	+−	−−
++	−+	+−	−+	−−	−+
++	−+	−+	++	++	−+

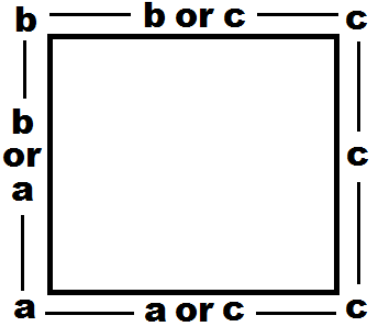
setting up the algebra

replace each entry by its label

b	b	c	b	c	c
b	c	b	c	a	c
b	a	c	b	b	c
a	c	b	c	c	c
a	c	c	a	a	c

setting up the algebra

note new boundary conditions

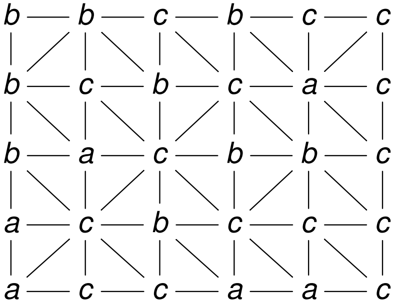


pass to the dual subdivision

b	*b*	*c*	*b*	*c*	*c*
b	*c*	*b*	*c*	*a*	*c*
b	*a*	*c*	*b*	*b*	*c*
a	*c*	*b*	*c*	*c*	*c*
a	*c*	*c*	*a*	*a*	*c*

setting up the algebra

triangulate each square *either* way



the oriented boundary map

Let E be the abelian group generated by the ordered pairs $\langle x, y \rangle$ where $x, y \in \{a, b, c\}$, subject to the relations

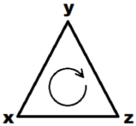
$$\langle x, y \rangle = -\langle y, x \rangle$$

$$\langle x, x \rangle = 0$$

for all $x, y \in \{a, b, c\}$.

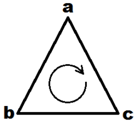
the oriented boundary map

Orient all triangles in the labeled planar simplicial complex compatibly (say, clockwise). Let ∂ be the map from labeled, oriented triangles to E defined by

$\partial\Big($ 

$\Big) = \langle x, y \rangle + \langle y, z \rangle + \langle z, x \rangle$

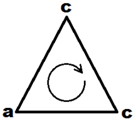
$\partial\Big($



$\Big) = \langle a, c\rangle + \langle c, b\rangle + \langle b, a\rangle$

$$= -\langle a, b\rangle - \langle b, c\rangle - \langle c, a\rangle$$

$\partial\Big($



$\Big) = \langle a, c\rangle + \langle c, c\rangle + \langle c, a\rangle$

$$= \langle a, c\rangle + 0 - \langle a, c\rangle = 0$$

labeled boundaries: the key property

A triangle is *well-labeled* if all three labels a, b, c show up on its vertices. It is *positive well-labeled* if a, b, c occur clockwise and *negative well-labeled* if they occur in the other orientation.

Lemma. Let T be a labeled triangle.

$\partial(T) = 0$ unless T is well-labeled
$\partial(T) = +\big[\langle a, b\rangle + \langle b, c\rangle + \langle c, a\rangle\big]$ if T is positive well-labeled
$\partial(T) = -\big[\langle a, b\rangle + \langle b, c\rangle + \langle c, a\rangle\big]$ if T is negative well-labeled.

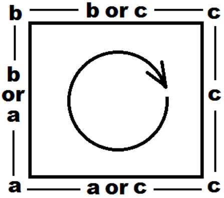
proof of the sign pattern theorem in two dimensions

Let's return to the labeled simplicial complex, with its $2(n-1)(m-1)$ triangles, obtained from the $n \times m$ sign matrix. Let w^+ and w^- denote the number of positive resp. negative well-labeled triangles it contains. Let's evaluate the sum S of the formal boundaries of triangles in two ways. By the lemma

$$S = \sum_{T \in \text{triangles}} \partial(T) = (w^+ - w^-)\big[\langle a, b\rangle + \langle b, c\rangle + \langle c, a\rangle\big]$$

proof of the sign pattern theorem in two dimensions

On the other hand, since the complex is oriented, the interior edges cancel and the sum equals the sum of oriented edges along the external boundaries.



The right-hand edge contributes 0. Apply the one-dimensional case of the sign pattern theorem to the other edges to see that their contribution is $\langle a, b\rangle$ resp. $\langle b, c\rangle$ resp. $\langle c, a\rangle$.

proof of the sign pattern theorem in two dimensions

$$S = \sum_{T \in \text{triangles}} \partial(T) = (w^+ - w^-)\left[\langle a, b\rangle + \langle b, c\rangle + \langle c, a\rangle\right]$$

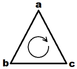
$$S = \langle a, b\rangle + \langle b, c\rangle + \langle c, a\rangle$$

$$w^+ - w^- = 1$$

- the number of triangles with all vertices labeled different, clockwise, is one more than the number of triangles with all vertices labeled different, counterclockwise
- can always find all four symbols $+ \; - \; + \; -$ within three corner-adjacent entries
- for any one of the four sign combinations, there are three corner-adjacent entries containing *that* specific sign combination, as well as all four symbols.

algebraic topological interpretation

• The triangulated sign matrix M, vertex-labeled with a, b, c, defines simplicial map to the triangle T

$M \to$ 

that takes the boundary ∂M of M to the boundary ∂T of T

• The map of pairs $(M, \partial M) \to (T, \partial T)$ induces a homomorphism of long exact sequences

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H_2(M) & \longrightarrow & H_2(M, \partial M) & \longrightarrow & H_1(\partial M) & \longrightarrow & H_1(M) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H_2(T) & \longrightarrow & H_2(T, \partial T) & \longrightarrow & H_1(\partial T) & \longrightarrow & H_1(T) & \longrightarrow & \dots \end{array}$$

algebraic topological interpretation

$$\begin{array}{ccccccccc} \dots \longrightarrow & H_2(M) & \longrightarrow & H_2(M, \partial M) & \longrightarrow & H_1(\partial M) & \longrightarrow & H_1(M) & \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \dots \longrightarrow & H_2(T) & \longrightarrow & H_2(T, \partial T) & \longrightarrow & H_1(\partial T) & \longrightarrow & H_1(T) & \longrightarrow \dots \end{array}$$

$$\begin{array}{ccccccccc} \dots \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\alpha} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow \dots \\ & \downarrow & & \beta\downarrow & & \downarrow\alpha & & \downarrow & \\ \dots \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\alpha} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow \dots \end{array}$$

• middle square commutes

• under the above labeling conventions, each α is the identity map, whence so is β

• that means that the number of preimages of the triangle T (counted with orientation) is 1

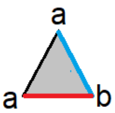
finding sign patterns by local rules

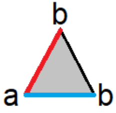
- input is the labeled, triangulated sign matrix
- deterministic algorithm selecting a sequence of edges (not necessarily a path)
- starts with the vertical edge from $(0, 0)$
- terminates with a well-labeled triangle

Let — denote the edge selected in the previous step and
— the edge selected in the current step.
There are three types of rules:

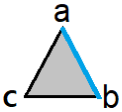
(1) 

and

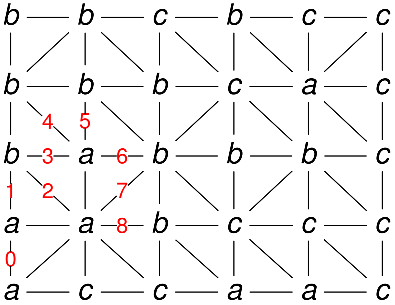

(2) 

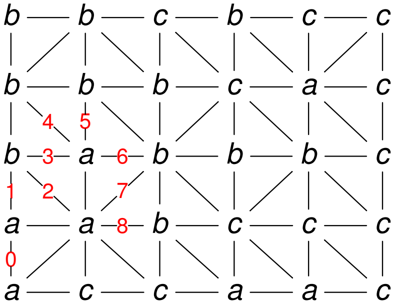
and


(and all their affine images)

(3) 

stop


b b c b c c
b b b c a c
4 5
b 3 a 6 b b b c
1 2 7
a a 8 b c c c
0
a c c a a c



Put the uniform distribution on the set of all $n \times n$ sign matrices and triangulate them in some uniform way (e.g. all diagonals running “north-east”).

Question Is the expected running time of this algorithm $O(n^d)$ for some $d < 2$?

Lefschetz fixed point theorem

Let X be a topological space that's homeomorphic to the geometric realization of a finite simplicial complex, with continuous self-map $f : X \to X$, k an arbitrary field.
f induces maps on homology $H_i(X, k) \to H_i(X, k)$.
Let $\mathrm{tr}(-)$ denote the trace of a linear map.

Define the *Lefschetz number*

$$L(f) = \sum_{i=0}^{\dim(X)} (-1)^i \,\mathrm{tr}\big(f|H_i(X,k)\big) \;\in\; k$$

Theorem (Lefschetz; Hopf) If $L(f) \neq 0$ then f has a fixed point.

$$L(f) = \sum_{i=0}^{\dim(X)} (-1)^i \operatorname{tr}\big(f|H_i(X,k)\big) \ \in \ k$$

If $L(f) \neq 0$ then f has a fixed point.

- One actually proves the contrapositive (*"if no fixed point then $L(f) = 0$"*) using simplicial approximation.
- One can show $L(f) \in \operatorname{im}(z)$ where $z : \mathbb{Z} \to k$ is the canonical map.
- Brouwer fixed point theorem follows by taking X to be $[0,1]^n$. In that case, $\operatorname{tr}(f|H_i) = 0$ if $i > 0$ and $\operatorname{tr}(f|H_0) = 1$.
- Restriction on the homeomorphism type of X is necessary; there are compact, contractible metric spaces X with continuous self-maps $f : X \to X$ that have no fixed point.

Lefschetz fixed point theorem for simplicial complexes

finite simplicial complex X
simplicial map $f : X \to X$
arbitrary field k
f induces maps on simplicial homology $f_i : H_i(X, k) \to H_i(X, k)$

Lefschetz number is defined the same way

$$L(f) = \sum_{i=0}^{\dim(X)} (-1)^i \operatorname{tr}\big(f|H_i(X, k)\big) \in k$$

Theorem If $L(f) \neq 0$ then X has an f-invariant face; that is, face S such that f restricted to S is a *bijection* $S \to S$.

Proof of the simplicial version of the Lefschetz fixed point theorem:

- apply Hopf's lemma to the chain complex (C_i, d_i) for computing simplicial homology

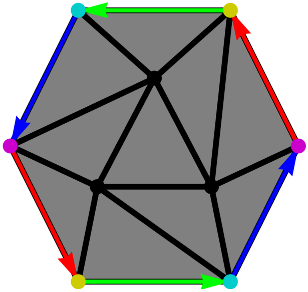
$$\sum_{i=0}^{\dim(X)} (-1)^i \operatorname{tr}(f|C_i) = \sum_{i=0}^{\dim(X)} (-1)^i \operatorname{tr}(f|H_i)$$

- if there is no invariant face, $\operatorname{tr}(f|C_i) = 0$ for all $0 \leqslant i \leqslant \dim(X)$.

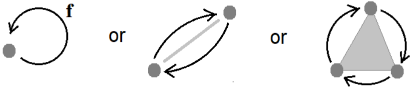
Also follows from the topological version (but doesn't seem to imply the topological version in full generality).

example

Triangulation T of real projective plane



For any simplicial map $f : T \to T$, there will exist



proof

For any even positive integer n,

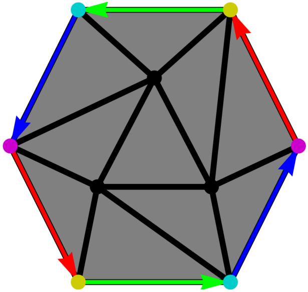
$$H_i(\mathbb{R}\mathrm{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i \text{ odd and } 0 < i < n \\ 0 & \text{otherwise} \end{cases}$$

$H_0(\mathbb{R}\mathrm{P}^n, \mathbb{Q}) = \mathbb{Q}$ and $H_i(\mathbb{R}\mathrm{P}^n, \mathbb{Q}) = 0$ for $i > 0$

$L(f) = 1$ for any continuous $f : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ with n even.

back to the example

This is the *minimal* triangulation of real projective plane



(“Half” an icosahedron; edges form a K_6.)

Trivially, for any simplicial map $f : T \to T$, there will exist an invariant K_i with $i \leqslant 6$.

Is there a less “highbrow” way to show the existence of an invariant i-face with $0 \leqslant i \leqslant 2$?

functors from graphs to simplicial complexes

For a graph G, consider the simplicial complex whose vertices are the vertices of G and whose faces are

(clique complex) the cliques of G

(independence complex) the anticliques of G

(neighborhood complex) sets of vertices of G that have a common neighbor

Determining the homology groups of these complexes (let alone the Lefschetz number induced by a self-map of the underlying graph!) can be hard.

Some combinatorial methods exist to prove that complexes of this type are *collapsible* (hence, topologically contractible).

Lefschetz-Hopf index formula: topological

Let X be homeomorphic to the geometric realization of a finite simplicial complex and $f : X \to X$ a continuous map. One can associate an *index* $I(f, p) \in \mathbb{Z}$ to each isolated fixed point p of f.

Theorem (Lefschetz; Hopf)

If f has finitely many fixed points p_i, $i \in J$ then

$$L(f) = \sum_{i=0}^{\dim(X)} (-1)^i \operatorname{tr}\big(f|H_i(X, \mathbb{Q})\big) = \sum_{i \in J} I(f, p_i) .$$

Lefschetz-Hopf index formula: simplicial (case of isolated fixed points)

Let X be a finite simplicial complex and $f : X \to X$ a simplicial map. There is a bijection between the isolated fixed points of $|f| : |X| \to |X|$ and the f-invariant faces S of X that are disjoint from other invariant faces (that is, if $T \neq S$ is also an invariant face then $S \cap T = \varnothing$).

The index of every isolated fixed point of $|f|$ is $+1$.

Hence, if all fixed points of $|f|$ are isolated (equivalently, if all f-invariant faces are disjoint) then their number equals

$$L(f) = \sum_{i=0}^{\dim(X)} (-1)^i \operatorname{tr}\big(f|H_i(X, \mathbb{Q})\big) \ .$$

Lefschetz-Hopf index formula: simplicial (general case)

Let X be a finite simplicial complex and $f : X \to X$ a simplicial map. Let F be the abstract simplicial complex whose vertices are the minimal (under inclusion) f-invariant faces of X; and if $m_1, m_2, \ldots, m_k$ are minimal f-invariant faces of X then $\{m_1, m_2, \ldots, m_k\}$ is a face of F iff $\bigcup_{i=1}^{k} m_i$ is a face of X.

The fixed point set of $|f|$ is a subcomplex of the first barycentric subdivision of $|X|$ and homeomorphic to $|F|$. Moreover,

$$L(f) = \sum_{i=0}^{\dim(X)} (-1)^i \operatorname{tr}\big(f|H_i(X, \mathbb{Q})\big) = \operatorname{eu}(F) = \sum_{C \in \pi_0(F)} \operatorname{eu}(C)$$

the Euler characteristic of the fixed point set of $|f|$ (equivalently, the sum of the Euler characteristics of its connected components).