The Fundamental Group of Abelian Varieties

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1 Tate Modules

Let \( A \) be a \( g \)-dimensional abelian variety over a field \( k \). Let \( \ell \) be a prime number different from \( \text{char}(k) \). Denote by \( k_s \) the separable closure of the field \( k \). For any nonnegative integer \( n \), we set a group

\[
A[\ell^n](k_s) := \text{Hom}(\mathbb{Z}/\ell^n\mathbb{Z}, A(k_s)).
\]

equipped with a natural Galois action by the absolute Galois group \( \text{Gal}(k_s/k) \). Hence, \( A[\ell^n](k_s) \) is the group of the \( \ell^n \)-torsion points in the group \( A(k_s) \) of \( k_s \)-points of \( A \). The multiplication by \( \ell \) morphism on \( A \) induces a homomorphism of groups

\[
\ell : A[\ell^{n+1}](k_s) \to A[\ell^n](k_s)
\]

which is \( \text{Gal}(k_s/k) \)-equivariant. Note that with these homomorphisms the collection \( \{A[\ell^n](k_s)\} \) forms a projective system of abelian groups with \( \text{Gal}(k_s/k) \)-action.

Definition 1.1. Let \( A \) be a \( g \)-dimensional abelian variety over a field \( k \), let \( k \subset k_s \) be a separable closure of the field \( k \), and let \( \ell \) be a prime number different from \( \text{char}(k) \). We define the Tate module \( T_\ell A \) of \( A \) to be

\[
T_\ell A = \varprojlim_{n \in \mathbb{Z}_{\geq 0}} \{A[\ell^n](k_s)\}.
\]

If \( \text{char}(k) = p > 0 \) then we define the Tate-\( p \)-module \( T_{p,\text{ét}} A \) to be

\[
T_{p,\text{ét}} A := \varprojlim_{n \in \mathbb{Z}_{\geq 0}} \{A[p^n](\bar{k})\}
\]

where the homomorphisms are multiplication by \( p \) and \( \bar{k} \) is the algebraic closure of the field \( k \).
2 The Fundamental Group of An Abelian Variety

Theorem 2.1 (Lang-Serre). Let $X$ be an abelian variety over a field $k$ with an identity $e_X \in X(k)$. Assume that $Y$ is a complete variety over the field $k$ with a $k$-point $e_Y \in Y(k)$. If $f : Y \to X$ is an étale covering with $f(e_Y) = e_X$ then $Y$ has the structure of an abelian variety such that the morphism $f$ is a separable isogeny.

In order to prove this theorem we need the following two lemmas.

Lemma 2.2. Let $X$ be a complete variety over a field $k$. Suppose given a $k$-point $e \in X(k)$ and a $k$-morphism $m : X \times X \to X$ such that $m(x, e) = x = m(e, x)$ for all $x \in X$. Then $X$ is an abelian variety with group law $m$ and the identity point $e$.

Proof. See [EvdGM, Chapter X, Proposition 10.34].

Lemma 2.3. Let $Z$ be a $k$-variety, let $Y$ be an integral $k$-scheme of finite type, and let $f : Y \to Z$ be a smooth proper morphism of $k$-schemes. If there exists a section $s : Z \to Y$ of the morphism $f$ then all fibres of $f$ are irreducible.

Proof. See [EvdGM, Chapter X, Lemma 10.35]

Proof of Theorem 2.1. By the Lemma 2.2, it suffices to construct a group law $m_Y : Y \times Y \to Y$. Let $\Gamma_X \subset X \times X \times X$ be the graph of the multiplication $m_X : X \times X \to X$, and let $\Gamma'_Y$ be the pullback of $\Gamma_X$ via the morphism $f \times f : Y \times Y \times Y \to X \times X \times X$.

We write $\Gamma_Y \subset \Gamma'_Y$ for the connected component containing the point $(e_Y, e_Y, e_Y)$. In the following, we show that the projection $q_{12} : \Gamma_Y \to Y \times Y$ from $\Gamma_Y$ to the first two factors is an isomorphism. In particular, we can define the desired group law by taking $m_Y := q_3 \circ q_{12}^{-1}$, where $q_3$ is the projection to the third factor.

There is a natural commutative diagram

\[
\begin{array}{ccc}
\Gamma_Y & \xrightarrow{q_{12}} & Y \times Y \\
\downarrow & & \downarrow f \times f \\
\Gamma_X & \xrightarrow{p_{12}} & X \times X.
\end{array}
\]

Recall that the graph $\Gamma'_Y$ is étale over $\Gamma_X$ and the connected component $\Gamma_Y$ is an open subset of $\Gamma_Y$, which implies that the left hand arrow is étale. Further, it follows that the morphism $q_{12}$ is étale since the bottom arrow is an isomorphism and $f \times f$ is étale. We claim that the finite étale morphism $q_{12}$ is an isomorphism. In fact, let $q_2 : \Gamma_Y \to Y$ be the composition of $q_{12}$ and the projection $p_2 : Y \times Y \to Y$ to the second factor. We have a section of the morphism $q_2$ induced by the identification $e_Y \times Y \sim Y \times Y$. 

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and the section \( s_1(e_Y, y) = (e_Y, y, y) \) of \( q_{12} \) over \( e_Y \times Y \) (Note that \( s_1 \) is a section that maps into \( \Gamma'_Y \) and the image \( s_1(e_Y \times Y) \cap \Gamma_Y \neq \emptyset \). Thus it is contained in \( \Gamma_Y \)). It follows from the Lemma 2.3 that any fiber of the morphism \( q_{12} \) is irreducible. We define \( Z := q_{12}^{-1}(e_Y) = q_{12}^{-1}(Y \times e_Y) \) the irreducible fiber of \( q_{12} \) over the origin \( e_Y \). Then it induces the pullback \( r : Z \rightarrow Y \times e_Y \) of the morphism \( q_{12} \), which is an finite etale morphism with the same degree of \( q_{12} \). On the other hand, we have a section \( s_2(y, e_Y) = (y, e_Y, y) \) of the morphism \( r \). It follows that \( r \) is an isomorphism and we conclude that \( q_{12} \) is an isomorphism too.

**Definition 2.4.** (Grothendieck) Let \( X \) be a scheme. Fix an algebraically closed field \( \Omega \) and a geometric point \( \bar{x} \). We define a functor

\[
F_{\bar{x}} : \text{F}_{\text{Et}/X} \rightarrow \text{Sets}
\]

from the category of finite étale morphisms over \( X \) to the category of sets by giving

\[
F_{\bar{x}}(f : Y \rightarrow X) = \{ y \in Y(\Omega) \mid f(y) = \bar{x} \}.
\]

In particular, assume that \( X \) is a connected locally noetherian scheme with a geometric point \( \bar{x} \). Then the étale fundamental group \( \pi_1^{\text{ét}}(X, \bar{x}) \) is defined to be the automorphism group of the functor \( F_{\bar{x}} \).

**Theorem 2.5.** (Grothendieck) Assume that \( X \) is a connected locally noetherian scheme with a geometric point \( \bar{x} \). Then \( \pi_1 := \pi_1^{\text{ét}}(X, \bar{x}) \) is a pro-finite group, and the functor \( F_{\bar{x}} \) induces an equivalence of categories

\[
\text{F}_{\text{Et}/X} \overset{\text{eq}}{\longrightarrow} \left( \text{finite } \pi_1 - \text{sets} \right)
\]

**Corollary 2.6.** Let \( A \) be an abelian variety over the field \( k \) with the origin \( e \in A(k) \), and let \( k_s \) be a separable closure of the field \( k \). Regard \( e \) as a geometric point, i.e., there is an algebraically closed field \( \Omega \) including \( k \). Then we have the canonical isomorphisms

\[
\pi_1^{\text{ét}}(A_{k_s}, e) \simeq \left\{ \begin{array}{ll}
\prod_{\ell} T_{\ell}A & \text{if } \text{char}(k) = 0, \\
T_{p,\text{ét}}A \times \prod_{\ell \neq p} T_{\ell}A & \text{if } \text{char}(k) = p > 0,
\end{array} \right.
\]

where the projective limit run over all maps \( A[nm](k_s) \rightarrow A[n](k_s) \) given by multiplication by \( m \), and where \( \ell \) runs over the prime numbers. Further, there exists a canonical isomorphism

\[
\pi_1^{\text{ét}}(A, e) \simeq \pi_1^{\text{ét}}(A_{k_s}, e) \rtimes \text{Gal}(k_s/k),
\]

where the Galois group acts on \( \pi_1^{\text{ét}}(A_{k_s}, e) \) through the action on the projective system \( \{ A[n](k_s) \}_{n \in \mathbb{Z}} \).

**Proof.** For simplicity, we denote by \( \pi = \pi_1^{\text{ét}}(X, \bar{x}) \) the étale fundamental group for any scheme \( X \) with a geometric point \( \bar{x} \). By the Theorem 2.5, we have

\[
\pi_1^{\text{ét}}(A_{k_s}, e) = \lim_{\text{proj}}(\pi/H)
\]
for all open subgroups, i.e., closed subgroups with finite index. It follows from the equivalence of categories in the Theorem 2.5 that each open subgroup \( H \) associates to an \( \acute{e}tale \) covering \( f_H : Y_H \to X \). By the Theorem of Lang-Serre, the \( k \)-variety \( Y_H \) is an abelian variety and \( f_H \) is a separable isogeny. Therefore, by [EvdGM, Proposition 5.6], it follows that the kernel \( \text{Ker}(f_H) \) is an \( \acute{e}tale \) \( k \)-group scheme and

\[
Y_H / \text{Ker}(f_H) \cong X.
\]

In particular, a separable isogeny is a Galois covering [EvdGM, Galois Covering 10.33]. Denote by \( I \) the set of isomorphism classes of separable isogenies \( f : Y \to X \) over \( X \). Two isogenies \( f : Y \to X \) and \( f' : Y' \to X \) are isomorphic if there exists an isomorphism of abelian varieties \( \alpha : Y \to Y' \) such that \( f' \circ \alpha = f \). We give a partial order on \( I \) by dominance. We say \( f \) dominates \( f' \), denote by \( f \geq f' \) if there exists a homomorphism of abelian varieties \( h : Y \to Y' \) such that \( f' \circ h = f \). In particular, the induced homomorphism of group schemes \( \text{Ker}(f) \to \text{Ker}(f') \) gives a projective system \( \{ \text{Ker}(f)(k_s) \}_{f \in I} \). It follows that

\[
\pi \simeq \lim_{\leftarrow \atop {f \in I}} \{ \text{Ker}(f)(k_s) \}.
\]

If \( n \) is a positive integer then \([n]_X : X \to X \) factors as

\[
X \xrightarrow{f} X/[n]_{\text{loc}} \xrightarrow{g} X
\]

where \( f \) is purely inseparable and \( g \) is separable. Here, the local group scheme \( X[n]_{\text{loc}} \) is the identity component of kernel \( X[n] \), which fits into a short exact sequence of group schemes

\[
1 \to X[n]_{\text{loc}} \to X[n] \to X[n]_{\text{et}} \to 1
\]

see [EvdGM, Proposition 4.45]. If \( \text{char}(k) = 0 \) or \( \text{char}(k) = p > 0 \) and \( p \nmid n \) then \( X[n]_{\text{loc}} = \{ \text{id} \} \) and \( [n]_X \) is separable. For the rest of the proof, we write \( g = [n]_{\text{sep}} \). Let \( I' \subset I \) be the subsets of isogenies \([n]_{\text{sep}}\) for \( n \in \mathbb{Z}_{\geq 1} \). Then \( I' \) is cofinal in \( I \), in fact, if \( f : Y \to X \) is a separable isogeny of degree \( d \), then there exist an isogeny \( g : X \to Y \) such that \( f \circ g = [d]_X \). It follows from [EvdGM, Corollary 5.8] that \([d]_{\text{sep}}\) dominates \( f \). Therefore, we have

\[
\pi \simeq \lim_{\rightarrow \atop {f \in I'}} \{ \text{Ker}(f)(k_s) \} = \lim_{\leftarrow \atop {n}} X[n](k_s).
\]

\[\square\]

3 Application

In this section, we show a fundamental relation between the \( \ell \)-adic cohomology of an abelian variety \( A \) and its Tate module \( T_\ell A \).

**Proposition 3.1.** Let \( A \) be an abelian variety over a field \( k \), and let \( k \subset k_s \) be a separable algebraic closure. Assume that \( \ell \) is a prime number with \( \ell \neq \text{char}(k) \). Then we have

\[
H^1(A_{k_s}, \mathbb{Z}_\ell) \simeq \text{Hom}(T_\ell A, \mathbb{Z}_\ell)
\]

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as $\mathbb{Z}_\ell$-modules with continuous action of $\text{Gal}(k_s/k)$.

**Proof.** In general, if $X$ is a complete variety over the $k$ with a geometric point $\bar{x}$, then there is an isomorphism

$$H^1(X_{k_s}, \mathbb{Z}_\ell) \cong \text{Hom}_{\text{cont}}(\pi_1^{\text{et}}(X_{k_s}, \bar{x}), \mathbb{Z}_\ell)$$

where $\text{Hom}_{\text{cont}}(,)$ means the continuous group homomorphisms. For the details, see Milne’s online notes [Mil80, Example 11.4]. Then the homomorphism

$$\text{Gal}(k_s/k) \to \text{Out}(\pi_1^{\text{et}}(X_{k_s}, \bar{x}))$$

induces a homomorphism

$$\text{Gal}(k_s/k) \to \text{Aut}(\pi_1^{\text{et}}(X_{k_s}, \bar{x})^{\text{ab}}).$$

It gives a continuous Galois action on $\text{Hom}_{\text{cont}}(\pi_1^{\text{et}}(X_{k_s}, \bar{x})^{\text{ab}}, \mathbb{Z}_\ell)$. In our case, it follows that

$$H^1(A_{k_s}, \mathbb{Z}_\ell) \cong \text{Hom}_{\text{cont}}(\pi_1^{\text{et}}(A_{k_s}, e)^{\text{ab}}, \mathbb{Z}_\ell) = \text{Hom}(\prod_\ell T_\ell A, \mathbb{Z}_\ell) = \text{Hom}(T_\ell A, \mathbb{Z}_\ell)$$

since the étale fundamental group $\pi_1^{\text{et}}(A_{k_s}, e) = \prod_\ell T_\ell A$ is abelian and a group homomorphism $\mathbb{Z}_{\ell'} \to \mathbb{Z}_\ell$ is trivial if $\ell' \neq \ell$. \qed

**References**
