AV SEMINAR NOTES 07/12

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These are notes for the abelian varieties seminar held at the UvA in 2016/2017. We discuss the Néron-Severi group of an abelian variety, and descent of line bundles along homomorphisms. We follow [EGM]. In most places where the exposition would be a verbatim copy of [EGM], we just refer to [EGM].
I thank Renjie Lyu and Raymond van Bommel for their help with the proof of the lemma in the appendix, Erik Visse for the interesting question, and Mingmin Shen for helping partially answer this question.

Contents

1 Néron-Severi group

Definition 1.1. If A is an abelian variety, we define the **Néron-Severi group** $\mathrm{NS}_{A/k}$ of A as the group of connected components of $\mathrm{Pic}_{A/k}$, i.e., it is defined by the exact sequence

$$0 \to \mathrm{Pic}^0_{A/k} \to \mathrm{Pic}_{A/k} \to \mathrm{NS}_{A/k} \to 0.$$

General group scheme theory shows that $\mathrm{NS}_{A/k}$ is locally of finite type over k.

For abelian varieties A and B, we use $\mathcal{H}\mathrm{om}_{\mathsf{AV}}(A, B)$ to denote the sheaf in abelian groups $T \mapsto \mathrm{Hom}_{\mathsf{Grp}}(A_T, B_T)$.
If $L \in \mathrm{Pic}(A_T)$, then $\Lambda(L)$ denotes the **Mumford line bundle** $m^*L \otimes p_1^*L^\vee \otimes p_2^*L^\vee$. This line bundle gives rise to a class

$$[\Lambda(L)] \in \mathrm{Pic}(A_T \times A_T)/p_2^* \mathrm{Pic}(A_T) = \mathrm{Pic}_{A_T/T}(A_T),$$

and hence a morphism $\varphi_L: A_T \to \mathrm{Pic}_{A_T/T}$.

Proposition 1.2. *There is a well-defined homomorphism* $\varphi: \mathrm{Pic}_{A/k} \to \mathcal{H}\mathrm{om}_{\mathsf{AV}}(A, A^\vee)$ *mapping the class of* $L \in \mathrm{Pic}(A_T)$ *in* $\mathrm{Pic}_{A/k}(T)$ *to* φ_L.

Proof. The proof of this can be found in [EGM, Lemma 7.11]. In proving that φ_L is a homomorphism for $L \in \mathrm{Pic}(A_T)$ with T reduced, they use Lemma 3.3 from the appendix. □

The rest of this section is devoted to the proof of the next result.

Theorem 1.3. *The sequence*

$$0 \to \mathrm{Pic}^0_{A/k} \to \mathrm{Pic}_{A/k} \to \mathcal{H}om_{\mathsf{AV}}(A, A^\vee)$$

is an exact sequence of fppf sheaves.

Remark 1.4. Zijian will show next time that the image of $\mathrm{Pic}_{A/k}$ consists of the self-dual morphisms, yielding an isomorphism $\mathrm{NS}_{A/k} \cong \mathcal{H}om_{\mathsf{AV}}{}^{\mathrm{symm}}(A, A^\vee)$.

Lemma 1.5. *The sheaf $\mathcal{H}om_{\mathsf{AV}}(A, B)$ is an étale commutative group scheme over k.*

Proof. Since abelian varieties are projective, FGA shows that $\mathcal{H}om(A, B)$ is a scheme locally of finite type over k. There is a cartesian diagram

$$\begin{array}{ccc} \mathcal{H}om_{\mathsf{AV}}(A, B) & \longrightarrow & \mathcal{H}om(A^2, B) \\ \big\downarrow & & \big\downarrow \Delta \\ \mathcal{H}om(A, B) & \xrightarrow[(\psi_1, \psi_2)]{} & \mathcal{H}om(A^2, B)^2 \end{array},$$

with $\psi_1(f)(a, a') := e_B$ and $\psi_2(f)(a, a') := f(a+a') - f(a) - f(a')$. Because the diagonal map is an immersion, and immersions are locally of finite type, $\mathcal{H}om_{\mathsf{AV}}(A, B)$ is a scheme locally of finite type over k.

For the étaleness, we reduce to the case $A = B$ by embedding $\mathcal{H}om_{\mathsf{AV}}(A, B)$ into $\mathcal{H}om_{\mathsf{AV}}(A \times B, A \times B)$ via $f \mapsto (a, b) \mapsto (a, f(b))$. By homogeneity, all we need to show is that $T_0\, \mathcal{H}om_{\mathsf{AV}}(A, A) = 0$, where 0 is the map $A \to A$, $a \mapsto 0$.

Let $A[\varepsilon]$ be the base change of A to $k[\varepsilon]$. Note that the underlying topological spaces of A and $A[\varepsilon]$ are the same, and that $\mathcal{O}_{A[\varepsilon]} = \mathcal{O}_A[\varepsilon]$. By definition, $T_0\, \mathcal{H}om_{\mathsf{AV}}(A, A)$ consists of the homomorphisms $f{:}A[\varepsilon] \to A[\varepsilon]$ such that

$$\begin{array}{ccc} A[\varepsilon] & \xrightarrow{f} & A[\varepsilon] \\ \big\downarrow & & \big\downarrow \\ A & \xrightarrow[0]{} & A \end{array}$$

commutes. Since the underlying topological spaces of A and $A[\varepsilon]$ coincide, this determines the underlying continuous map of f. Moreover, because f is a homomorphism it maps the identity section $e \in A[\varepsilon](k[\varepsilon])$ to itself, giving the commutativity of

$$\begin{array}{ccc} \mathcal{O}_A[\varepsilon] & \xrightarrow{f^\sharp} & 0_* \mathcal{O}_A[\varepsilon] \\ e^\sharp \big\downarrow & & \big\downarrow 0_* e^\sharp \\ e_* k[\varepsilon] & \text{———} & 0_* e_* k[\varepsilon] \end{array}.$$

The reader is invited to pretend that the line at the bottom is a long equals sign. The completeness of A, and the fact that $0 = e\pi$, where $\pi{:}A[\varepsilon] \to \operatorname{Spec}(k[\varepsilon])$ is the structure morphism, shows that the map on the right is an isomorphism, and hence that $f^\sharp$ is uniquely determined. □

Remark 1.6. Let k be algebraically closed (so that the group scheme $\mathcal{H}\mathrm{om}_{\mathsf{AV}}(A, A^\vee)$ is completely determined by its k-points, i.e., the group $\operatorname{Hom}(A, A^\vee)$), and let H be some suitable Weil cohomology. Erik asked if we could map $\operatorname{Hom}(A, A^\vee)$ to $H^2(A)$ in such a way that we recover c_1 when we restrict to $\mathrm{NS}(A) := \mathrm{NS}_{A/k}(k)$. Consider the composition

$$\operatorname{Hom}(A, A^\vee) \to H^{2g}(A \times A^\vee) \to H^{2g-1}(A^\vee) \otimes H^1(A) \to H^1(A) \otimes H^1(A) \to H^2(A),$$

where the first map sends f to its graph, the second is a projection onto a Künneth component, the third is the correspondence induced by the Poincaré bundle, and the fourth is the cup product. This gives us two maps $\operatorname{Pic}(A) \to H^2(A)$ (the second one being c_1).
Question: Do these two maps coincide?

Corollary 1.7. $\operatorname{Pic}^0_{A/k}$ *is contained in the kernel of* $\operatorname{Pic}_{A/k} \to \mathcal{H}\mathrm{om}_{\mathsf{AV}}(A, A^\vee)$.

Proof. $\mathcal{H}\mathrm{om}_{\mathsf{AV}}(A, A^\vee)$ is discrete (because $\mathcal{H}\mathrm{om}_{\mathsf{AV}}(A, A^\vee)$ is étale over k), $\operatorname{Pic}^0_{A/k}$ is connected, and φ is a homomorphism. □

Lemma 1.8. *Let* $L \in \operatorname{Ker}(\varphi)(k)$ *with* $L \not\cong \mathcal{O}_A$. *Then* $H^i(A, L) = 0$ *for all* i.

Proof. Since $\varphi_L = 0$, there holds $(-1)^*L \cong L^\vee$ [EGM, Lemma 7.16], so L has a global section if and only if $L^\vee$ does. It follows that if L is non-trivial, $H^0(A, L) = 0$.
We proceed by induction on i. From the commutative diagram of schemes

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A \times 0} & A \times A \\ & \searrow^{\mathrm{id}_A} & \downarrow m \\ & & A \end{array}$$

we get the commutative diagram

$$\begin{array}{ccc} H^i(A, L) & \longleftarrow & H^i(A \times A, m^*L) \\ & \nwarrow^{\mathrm{id}} & \uparrow \\ & & H^i(A, L) \end{array} \qquad (1)$$

The triviality of $\Lambda(L)$, Künneth, and induction show that

$$H^i(A \times A, m^*L) \cong H^i(A \times A, L \boxtimes L) \cong \bigoplus_{a+b=i} \Big(H^a(A, L) \otimes_k H^b(A, L) \Big) = 0,$$

which proves $H^i(A, L) = 0$ by Diagram (1). □

Proposition 1.9 ([EGM, Proposition 7.21])**.** *Assume k is algebraically closed, and let L be an ample line bundle on A. Then for any $M \in \mathrm{Ker}(\varphi)(k)$ there exists $x \in A(k)$ such that $M \cong L^\vee \otimes t_x^* L$.*

This proposition allows us to conclude the proof of Theorem 1.3.

Corollary 1.10. $\mathrm{Pic}^0_{A/k} = \mathrm{Ker}(\varphi)$

Proof. We can check this on geometric points, where it is a consequence of the proposition, since bundles of the form $L^\vee \otimes t_x^* L$ are degree 0. □

2 Theta groups and descent of line bundles

2.1 G-equivariant sheaves

Throughout this section X is a scheme over S, and G a group scheme over S acting on X.

Definition 2.1. Let $F \in \mathsf{QCoh}(X)$. A G**-action** on F consists of a collection of isomorphisms $\lambda_g : F_T \to \rho_g^* F_T$ in $\mathsf{QCoh}(X_T)$ for all $g \in G(T)$, where F_T is the pullback of F to X_T, and ρ_g is the action of g on X_T. For $g, g' \in G(T)$, we require that

$$\begin{array}{ccc} F_T & \xrightarrow{\lambda_{gg'}} & \rho_{gg'}^* F_T \\ {\scriptstyle \lambda_{g'}}\downarrow & & \downarrow \\ \rho_{g'}^* F_T & \xrightarrow[\rho_{g'}^* \lambda_g]{} & \rho_{g'}^* \rho_g^* F_T \end{array}$$

commutes. Such a pair (F, λ) is called a G**-equivariant sheaf**, and with the obvious notion of morphisms these form a category $\mathsf{QCoh}_G(X)$ over $\mathsf{QCoh}(X)$.

Remark 2.2. It is maybe not immediately apparent why data as above should be called G-actions, or G-equivariant sheaves. In terms of the sheaf $\overline{F} \in \mathsf{Sh}\left((\mathsf{Sch}/S)_{\mathrm{fppf}}\right)$ associated to F, giving a G-action on F is the same as giving an action $G \times \overline{F} \to \overline{F}$ such that $\overline{F} \to X$ is G-equivariant. This straightforward exercise is left to the reader.

Theorem 2.3 (Descent along torsors)**.** *If $f : X \to Y$ is a G-invariant map in Sch/S then for any $F \in \mathsf{QCoh}(Y)$, the pullback f^*F comes with a canonical G-action, giving a functor $\mathsf{QCoh}(Y) \to \mathsf{QCoh}_G(X)$. When $X \to Y$ is a G-torsor in the fppf topology, this functor is an equivalence, which restricts to an equivalence between lffr sheaves on Y, and G-equivariant lffr sheaves on X.*

Proof sketch. A G-action on a sheaf translates directly to a descent datum on that sheaf using the isomorphism $G \times X \to X \times_Y X$, $(g, x) \mapsto (gx, x)$. Then, since torsors are local epimorphisms, faithfully flat descent allows us to conclude. □

2.2 Theta groups

In this section we generalize the construction Raymond used to argue that $\mathrm{Pic}^0_{A/k}$ is isomorphic to $\underline{\mathrm{Ext}}^1(\mathbb{G}_m, A)$, and use it to study when a line bundle descends along a homomorphism.

Definition 2.4. For a line bundle $L \in \mathrm{Pic}(A)$, we define the **theta group of** L to be the functor $\mathcal{G}(L){:}(\mathsf{Sch}/k)^{\mathrm{opp}} \to \mathsf{Grp}$ given by mapping T/k to

$$\{(a,\lambda) \mid a \in A(T),\ \lambda{:}L_T \to t_a^* L_T \text{ an isomorphism}\}\,.$$

The group law on this set is given by

$$(a,\lambda)(b,\mu) = (a+b, (t_b^*\lambda)\mu)\,.$$

Remarks 2.5.

1. For a subgroup $H \hookrightarrow A$, it follows immediately from the definitions that there is a bijective correspondence

$$\{H\text{-actions on } L\} \longleftrightarrow \left\{ \text{lifts} \quad \begin{array}{ccc} & & \mathcal{G}(L) \\ & \nearrow & \downarrow \\ H & \longrightarrow & A \end{array} \right\}.$$

2. There is a homomorphism $\mathbb{G}_m \to \mathcal{G}(L)$, mapping $t \in \mathbb{G}_m(T)$ to the automorphism of L_T given by multiplying by t. The homomorphism $\mathcal{G}(L) \to A$ factors through $K(L)$, by definition of $K(L)$.

Lemma 2.6. *There is a short exact sequence*

$$0 \to \mathbb{G}_{m,k} \to \mathcal{G}(L) \to K(L) \to 0$$

of sheaves on the big Zariski site of S. In particular, $\mathcal{G}(L)$ is a group k-scheme.

Proof. Very similar to what Raymond did. Omitted. □

Note that $\mathcal{G}(L)$ is not necessarily commutative. Its failure to be so is measured by the morphism $[,]{:}\,\mathcal{G}(L) \times \mathcal{G}(L) \to \mathcal{G}(L)$, $(g,h) \mapsto ghg^{-1}h^{-1}$. Since $K(L)$ is commutative, this actually lands in $\mathbb{G}_m$, and since $\mathbb{G}_m$ is central in $\mathcal{G}(L)$, it factors through a map $e^L{:}K(L) \times K(L) \to \mathbb{G}_m$.

Definition 2.7. The pairing $e^L{:}K(L) \times K(L) \to \mathbb{G}_m$ constructed above is called the **commutator pairing** of L.

Remark 2.8. When L is a degree 0 line bundle, $K(L) = A$, so that e^L must be constant (being a morphism from a complete to an affine variety). It follows that $\mathcal{G}(L)$ is commutative, and it can be shown that the extension in the lemma coincides with the extension Raymond constructed in his talk to give an isomorphism $A^\vee \to \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$.

Proposition 2.9. *If $f{:}A \to B$ is an isogeny of abelian varieties over an algebraically closed field k, then $L \in \mathrm{Pic}(A)$ descends to B if and only if $\mathrm{Ker}(f)$ is a subgroup of $K(L)$ which is e^L-isotropic.*

Proof. If $\mathrm{Ker}(f)$ is an e^L-isotropic subgroup of $K(L)$, then we can pull the extension of Lemma 2.6 back to $\mathrm{Ker}(f)$ to get an extension of $\mathrm{Ker}(f)$ by $\mathbb{G}_m$. This extension is commutative because $\mathrm{Ker}(f)$ is e^L-isotropic. Since k is algebraically closed and $\mathrm{Ker}(f)$ finite, $\mathrm{Ext}^1(\mathrm{Ker}(f), \mathbb{G}_m) = 0$, showing that the sequence splits, yielding a lift of $\mathrm{Ker}(f) \to A$ to $\mathcal{G}(L)$. Now L descends by the fact that $A \to B$ is a $\mathrm{Ker}(f)$-torsor, Remark 2.5.1, and descent along torsors.
Conversely, if L descends, we get a lift $\mathrm{Ker}(f) \to \mathcal{G}(L)$, which shows that $\mathrm{Ker}(f)$ is a subgroup of $K(L)$ (because $\mathcal{G}(L) \to A$ factors through $K(L)$), and that e^L is isotropic on $\mathrm{Ker}(f)$ (because we can use the section to get commuting lifts to $\mathcal{G}(L)$ of any two elements of $\mathrm{Ker}(f)$). □

Proposition 2.10 (Properties of the commutator pairing)**.**

1. *If $f{:}A \to B$ is a homomorphism, and $L \in \mathrm{Pic}(B)$, then $e^L(f, f) = e^{f^*L}$ on $f^{-1}(K(L))$.*
2. *If $L, M \in \mathrm{Pic}(A)$, then $e^{L\otimes M} = e^L e^M$ on $K(L) \cap K(M)$.*
3. *For $n \in \mathbb{Z}$, $x, ny \in K(L)(T)$ we have $e^L(x, ny) = e^{L^{\otimes n}}(x, y)$.*

Proof. The proof of this proposition is trivial if one uses the following geometric characterization of points of $\mathcal{G}(L)$. The line bundle L on A corresponds to a geometric line bundle $\mathbb{L} := \mathrm{Spec}_A(\mathrm{Sym}(L^\vee)) \to A$ (see [GW, Section 11.3]), and a morphism $L_T \to t_a^* L_T$ is the same as a fiberwise linear morphism $\mathbb{L}_T \to \mathbb{L}_T$ such that

$$\begin{array}{ccc} \mathbb{L}_T & \longrightarrow & \mathbb{L}_T \\ \downarrow & & \downarrow \\ A_T & \xrightarrow[t_a]{} & A_T \end{array}$$

commutes. □

3 Appendix

Definition 3.1. A morphism $f{:}X \to Y$ of schemes is called **schematically dominant** if $f^\sharp{:}\, \mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective.

The following lemma is an easy exercise. Most of it can be found in [EGAIV, §11.10].

Lemma 3.2.

1. *A schematically dominant closed immersion is an isomorphism.*

2. *If the composition gf is schematically dominant, then g is.*
3. *If morphisms $f, g{:}X \to Y$ of T-schemes are equalized by a schematically dominant morphism, and Y is separated over T, then $f = g$.*
4. *Schematic dominance is preserved under faithfully flat base change.*
5. *A scheme T is reduced if and only if the natural morphism*

$$\pi{:}S := \bigsqcup_{t \in T} \operatorname{Spec}(\Bbbk(t)) \longrightarrow T$$

is schematically dominant.

Proof. For the fifth point, let $\operatorname{Spec}(R) = U$ be an open affine in T. Then the map

$$\pi_U^\sharp{:}\Gamma(U, \mathcal{O}_T) = R \longrightarrow \Gamma(U, \pi_* \mathcal{O}_S) = \prod_{t \in U} \Bbbk(t) = \prod_{\mathfrak{p} \in \operatorname{Spec}(R)} R_\mathfrak{p}/\mathfrak{p}$$

has kernel $\operatorname{nil}(R) = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p}$, which vanishes for all such U if and only if T is reduced. □

The proof of Proposition 1.2 uses the following fact.

Lemma 3.3. *Let T be a reduced scheme, X a faithfully flat T-scheme, Y a separated T-scheme, and $f, g{:}X \to Y$ morphisms of T-schemes. If f and g agree on the fibers of $X \to T$, then $f = g$.*

Proof. Since T is reduced, the map $\bigsqcup_{t \in T} \operatorname{Spec}(\Bbbk(t)) \to T$ is schematically dominant. Schematic dominance is preserved under faithfully flat base change, so $\pi{:}\bigsqcup_{t \in T} X_t \to X$ is schematically dominant, where X_t denotes the fiber of $X \to T$ over t. By assumption π equalizes f and g, so Point 3 of Lemma 3.2 ends the proof. □

References

[EGAIV] **J. Dieudonné, A. Grothendieck**. *Eléments de Géométrie Algébrique IV.* Publ. Math. IHÉS, **20, 24, 28, 32** (1964–1967).

[EGM] **B. Edixhoven, G. van der Geer, B. Moonen**. *Abelian Varieties.* Book project, available on Ben Moonen's home page.

[GW] **U. Görtz, T. Wedhorn**. *Algebraic Geometry I.* Vieweg+Teubner, 2010.