The Variable Hierarchy for the Games μ-Calculus (abstract)

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An extended version of this abstract is the reference [2].
Parity games are combinatorial representations of closed Boolean μ-terms. By adding to them draw position (or free variables), A.Arnold and L. Santocanale [1] have structured parity games into *the games μ-calculus*. In other words, the authors defined substitution, least and greatest fixed-point operators, as usual for μ-calculi. The canonical interpretation of this μ-calculus is over the class of all complete lattices. As done by Berwanger et al.[3] for the propositional μ-calculus it is possible to classify parity games into *levels of a hierarchy* according to the number of fixed point variables. We ask whether this hierarchy collapses w.r.t the canonical interpretation. We answer this question negatively, that is the Theorem bellow.
A preorder on the collection of labeled parity games, noted hereby $\mathcal{G}$, is described by means of Mediator-Opponent game $\langle G, H\rangle$. We declare that $G \leq H$ iff Mediator has a winning strategy in $\langle G, H\rangle$ – and write $G \sim H$ if $G \leq H$ and $H \leq G$. The relation $\leq$ has been proved *sound and complete* w.r.t the canonical interpretation in any complete lattice [5]. The main tools used to capture the combinatorial essence of the variable hierarchy are two graph metrics: the *feedback* and the *entanglement*. Asking whether a parity game G is semantically equivalent to a μ-term with at most n-variables amounts to asking whether G belongs to the level $\mathcal{L}_n$ defined as follows: $\mathcal{L}_n = \{\, G \in \mathcal{G} \mid G \sim H \text{ for some } H \in \mathcal{G} \text{ s.t. } \mathcal{E}(H) \leq n \,\}$, where $\mathcal{E}(H)$ denotes the entanglement of the underlying graph of H. The variable hierarchy being made up of the levels $\mathcal{L}_n$, the hierarchy problem can be formalized as follows: *is there a constant $k \geq 0$ such that for all $n \geq k$, we have $\mathcal{L}_k = \mathcal{L}_n$*? We answer this question negatively by constructing, for each $n \geq 1$, a parity game G_n with these properties: (i) G_n unravels to a tree with back-edges (i.e. μ-term) of feedback n, showing that $G_n \in \mathcal{L}_n$, (ii) G_n is semantically equivalent to no game in $\mathcal{L}_{n-3}$. Therefore, we prove that the inclusions $\mathcal{L}_{n-3} \subseteq \mathcal{L}_n$, $n \geq 3$, are strict. The games G_n strengthen the notion of *synchronizing games* from [4] to the context of the variable hierarchy. We prove that the syntactical structure of a game H, which is semantically equivalent to G_n, resembles that of G_n: every move (edge) in G_n can be simulated by a non empty finite sequence of moves (a path) of H; if two paths simulating distinct edges do intersect, then the edges do intersect as well. We formalize such situation within the notion of *$\star$-weak simulation*. The main first result is that if there is a weak simulation of G by H, then $\mathcal{E}(G) \leq \mathcal{E}(H)+2$, and moreover if G is a strongly synchronizing game, and if $H \in \mathcal{G}$ is such that $G \leq H \leq G$, then there is a $\star$-weak simulation of G by H. Summing up these observations we obtain our main theorem.

Theorem 1. *For $n \geq 3$, the inclusions $\mathcal{L}_{n-3} \subseteq \mathcal{L}_n$ are strict. Therefore the variable hierarchy for the games μ-calculus is infinite.*

A relevant question is about interpreting a μ-calculus into an other one, this should be a bridge relating results obtained in different μ-calculi and eventually relating the results on the μ-calculus of parity games and those of the μ-calculus of modal logics.

References

1. André Arnold and Luigi Santocanale. Ambiguous classes in the games μ-calculus hierarchy. In *FOSSACS 2003*, volume 2620 of *Lecture Notes in Comput. Sci.*, pages 70–86. Springer, Berlin, 2003.
2. Walid Belkhir and Luigi Santocanale. The variable hierarchy for the games μ-calculus. Submitted to the conference LICS 2008, http://hal.archives-ouvertes.fr/hal-00178806/, October 2007.
3. Dietmar Berwanger and Giacomo Lenzi. The variable hierarchy of the μ-calculus is strict. In *STACS 2005*, volume 3404 of *LNCS*, pages 97–109. Springer-Verlag, 2005.
4. Luigi Santocanale. The alternation hierarchy for the theory of μ-lattices. *Theory and Applications of Categories*, 9:166–197, January 2002.
5. Luigi Santocanale. Free μ-lattices. *Journal of Pure and Applied Algebra*, 168(2-3):227–264, March 2002.