The Variable Hierarchy for the Games $\mu$-Calculus (abstract)

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An extended version of this abstract is the reference [2].

Parity games are combinatorial representations of closed Boolean $\mu$-terms. By adding to them draw position (or free variables), A.Arnold and L. Santocanale [1] have structured parity games into the games $\mu$-calculus. In other words, the authors defined substitution, least and greatest fixed-point operators, as usual for $\mu$-calculi. The canonical interpretation of this $\mu$-calculus is over the class of all complete lattices.

As done by Berwanger et al.[3] for the propositional $\mu$-calculus it is possible to classify parity games into levels of a hierarchy according to the number of fixed point variables. We ask whether this hierarchy collapses w.r.t the canonical interpretation. We answer this question negatively by constructing, for each $n \geq k$, a parity game $G_n$ which is semantically equivalent to no game in $L_n$. Therefore, we prove that the inclusions $L_{n-3} \subseteq L_n$, $n \geq 3$, are strict. The games $G_n$ strengthen the notion of synchronizing games from [4] to the context of the variable hierarchy. We prove that the syntactical structure of a game $H$, which is semantically equivalent to $G_n$, resembles that of $G_n$: every move (edge) in $G_n$ can be simulated by a non empty finite sequence of moves (a path) of $H$; if two paths simulating distinct edges do intersect, then the edges do intersect as well. We formalize such situation within the notion of $*$-weak simulation. The main first result is that if there is a weak simulation of $G$ by $H$, then $\mathcal{E}(G) \leq \mathcal{E}(H) + 2$, and moreover if $G$ is a strongly synchronizing game, and if $H \in \mathcal{G}$ is such that $G \leq H \leq G$, then there is a $*$-weak simulation of $G$ by $H$. Summing up these observations we obtain our main theorem.

**Theorem 1.** For $n \geq 3$, the inclusions $L_{n-3} \subseteq L_n$ are strict. Therefore the variable hierarchy for the games $\mu$-calculus is infinite.

A relevant question is about interpreting a $\mu$-calculus into an other one, this should be a bridge relating results obtained in different $\mu$-calculi and eventually relating the results on the $\mu$-calculus of parity games and those of the $\mu$-calculus of modal logics.

**References**