ICPDL with fixed points and nominals

Stefan Göller and Markus Lohrey

Institute for Computer Science
University of Leipzig, Germany

We introduce the logic $\mu$-ICPDL$_{\text{Nom}}$ which subsumes the modal $\mu$-calculus and PDL with intersection and converse of programs (ICPDL) and allows the usage of nominals. Let $P$ be a countable set of atomic propositions, let $A$ be a countable set of atomic programs. Moreover, let $X$ denote a countable set of fixed point variables and $\text{Nom}$ a countable set of nominals. The set of formulas $\Phi$ and the set of programs $\Pi$ of $\mu$-ICPDL$_{\text{Nom}}$ are the smallest sets such that: (i) $P \cup X \cup \text{Nom} \subseteq \Phi$, (ii) if $\varphi \in \Phi$, then $\neg \varphi \in \Phi$, (iii) if $\pi \in \Pi$ and $\varphi \in \Phi$, then $\langle \pi \rangle \varphi \in \Phi$, (iv) if $X \in X$, $\varphi \in \Phi$, and every free occurrence of $X$ in $\varphi$ is within an even number of negations, then $\mu X. \varphi \in \Phi$, (v) if $p \in \text{Nom}$ and $\varphi \in \Phi$, then $\emptyset_p \varphi \in \Phi$, and (v) $\emptyset \in \Phi$.

A Kripke structure with respect to a finite set of nominals $N \subseteq \text{Nom}$ is a tuple $K = (W, \{-a\} \mid a \in A, \{W_p \mid p \in P \cup N\}, \rho)$, where (i) $W$ is a set of worlds, (ii) $\rightarrow_a \subseteq W \times W$ is a binary relation for each $a \in A$, (iii) $W_p \subseteq W$ for each $p \in P \cup N$ and $|W_p| = 1$ for each $p \in N$, and (iv) $\rho : X \rightarrow 2^W$ is an evaluation (of fixed point variables). For a subset $V \subseteq W$, a fixed point variable $X \in X$ and an evaluation $\rho$, we denote by $\rho |X \rightarrow V$ the evaluation that is defined as $\rho |X \rightarrow V(Y) = V$ if $Y = X$ and $\rho(Y)$ else. Extending the latter notation to Kripke structures, we define $K[X \rightarrow V] = (W, \{-a\} \mid a \in A, \{W_p \mid p \in P \cup N\}, \rho |X \rightarrow V)$.

Making use of the Knaster-Tarski fixed point theorem, we can define the semantics of $\mu$-ICPDL$_{\text{Nom}}$. Let $K = (W, \{-a\} \mid a \in A, \{W_p \mid p \in P \cup N\}, \rho)$ be a Kripke structure. Assuming that only nominals from $N$ are used, we define for each $\varphi \in \Phi$ a subset $[\varphi]_K \subseteq W$ and for each $\pi \in \Pi$ a binary relation $[\pi]_K \subseteq W \times W$ as follows:

\[
\begin{align*}
[\varphi]_K &= W_p \text{ for } p \in P \cup N & [a]_K &= \neg a \text{ for } a \in A \\
[w \cup \varphi]_K &= \rho(X) \text{ for } X \in X & [\{x, y\} \mid x, y \in [\varphi]_K] &= \{(y, x) \mid x \rightarrow_a y\} \text{ for } a \in A \\
[\neg \varphi]_K &= W \setminus [\varphi]_K & [\varphi | x, y\} \mid x, y \in [\varphi]_K] &= \{(w, w) \mid w \in [\varphi]_K\} \\
[\langle \pi \rangle \varphi]_K &= \{x \mid \exists y : (x, y) \in [\pi]_K \land y \in [\varphi]_K\} & [\pi_1 \text{ op } \pi_2]_K &= [\pi_1]_K \text{ op } [\pi_2]_K \text{, op } \in \{\cup, \neg, \circ\} \\
[\mu X. \varphi]_K &= \text{lpf}(U \mapsto [\varphi]_K[X \rightarrow U]) & [\pi_1 \text{ op } \pi_2]_K &= [\pi_1]_K \text{ op } [\pi_2]_K, \text{ op } \in \{\cup, \neg, \circ\} \\
[\emptyset \pi \varphi]_K &= \begin{cases} W & \text{if } W_p \subseteq [\varphi]_K \text{ for } p \in N \\ \emptyset & \text{else} \end{cases}
\end{align*}
\]

where $\text{lpf}(U \mapsto [\varphi]_K[X \rightarrow U])$ denotes the least fixed point of the monotone function $U \mapsto [\varphi]_K[X \rightarrow U]$. A Kripke structure $K$ is a model for a formula $\varphi$ if for some world $w$ of $K$ we have $w \in [\varphi]_K$. A formula $\varphi$ is satisfiable if it has a model. The following two results extend corresponding theorems from [1].

**Theorem 1.** Let $N \subseteq \text{Nom}$ be a finite set of nominals. Every satisfiable $\mu$-ICPDL$_{\text{Nom}}$ formula that uses only nominals from $N$ has a countable model of tree width at most $|N| + 2$.

Combining this model theoretic result with automata theoretic techniques (translation into two-way alternating tree automata with a parity acceptance condition), we can show the following theorem.

**Theorem 2.** Satisfiability in $\mu$-ICPDL$_{\text{Nom}}$ is $\text{2EXPTIME}$-complete.

**References**