ICPDL with fixed points and nominals

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We introduce the logic μ-ICPDL$_{\texttt{Nom}}$ which subsumes the modal μ-calculus and PDL with intersection and converse of programs (ICPDL) and allows the usage of nominals. Let $\mathbb{P}$ be a countable set of *atomic propositions*, let $\mathbb{A}$ be a countable set of *atomic programs*. Moreover, let $\mathbb{X}$ denote a countable set of *fixed point variables* and $\texttt{Nom}$ a countable set of *nominals*. *The set of formulas* Φ and *the set of programs* Π of μ-ICPDL$_{\texttt{Nom}}$ are the smallest sets such that: (i) $\mathbb{P} \cup \mathbb{X} \cup \texttt{Nom} \subseteq \Phi$, (ii) if $\varphi \in \Phi$, then $\neg\varphi \in \Phi$, (iii) if $\pi \in \Pi$ and $\varphi \in \Phi$, then $\langle\pi\rangle\varphi \in \Phi$, (iv) if $X \in \mathbb{X}$, $\varphi \in \Phi$, and every free occurrence of X in φ is within an even number of negations, then $\mu X.\varphi \in \Phi$, (v) if $p \in \texttt{Nom}$ and $\varphi \in \Phi$, then $@_p\varphi \in \Phi$, (vi) $\mathbb{A} \cup \{\overline{a} \mid a \in \mathbb{A}\} \subseteq \Pi$, (vii) $\{\varphi? \mid \varphi \in \Phi\} \subseteq \Pi$, (viii) if $\pi_1, \pi_2 \in \Pi$, then $\pi_1^* \in \Pi$ and π_1 op π_2 for each op $\in \{\cup, \cap, \circ\}$. A *Kripke structure with respect to a finite set of nominals* $N \subseteq \texttt{Nom}$ is a tuple $K = (W, \{\rightarrow_a \mid a \in \mathbb{A}\}, \{W_p \mid p \in \mathbb{P} \cup N\}, \rho)$, where (i) W is a set of *worlds*, (ii) $\rightarrow_a \subseteq W \times W$ is a binary relation for each $a \in \mathbb{A}$, (iii) $W_p \subseteq W$ for each $p \in \mathbb{P} \cup N$ and $|W_p| = 1$ for each $p \in N$, and (iv) $\rho : \mathbb{X} \rightarrow 2^W$ is an *evaluation* (of fixed point variables). For a subset $V \subseteq W$, a fixed point variable $X \in \mathbb{X}$ and an evaluation ρ, we denote by $\rho[X \rightarrow V]$ the evaluation that is defined as $\rho[X \rightarrow V](Y) = V$ if $Y = X$ and $\rho(Y)$ else. Extending the latter notation to Kripke structures, we define $K[X \rightarrow V] = (W, \{\rightarrow_a \mid a \in \mathbb{A}\}, \{W_p \mid p \in \mathbb{P} \cup N\}, \rho[X \rightarrow V])$. Making use of the Knaster-Tarksi fixed point theorem, we can define the semantics of μ-ICPDL$_{\texttt{Nom}}$. Let $K = (W, \{\rightarrow_a \mid a \in \mathbb{A}\}, \{W_p \mid p \in \mathbb{P} \cup N\}, \rho)$ be a Kripke structure. Assuming that only nominals from N are used, we define for each $\varphi \in \Phi$ a subset $[\![\varphi]\!]_K \subseteq W$ and for each $\pi \in \Pi$ a binary relation $[\![\pi]\!]_K \subseteq W \times W$ as follows:

$$\begin{array}{rcl}
[\![p]\!]_K &=& W_p \quad \text{for } p \in \mathbb{P} \cup N \\
[\![X]\!]_K &=& \rho(X) \text{ for } X \in \mathbb{X} \\
[\![\neg\varphi]\!]_K &=& W \setminus [\![\varphi_K]\!] \\
[\![\langle\pi\rangle\varphi]\!]_K &=& \{x \mid \exists y : (x,y) \in [\![\pi]\!]_K \wedge y \in [\![\varphi]\!]_K\} \\
[\![\mu X.\varphi]\!]_K &=& \mathbf{lfp}(U \mapsto [\![\varphi]\!]_{K[X \rightarrow U]}) \\
[\![@_p\varphi]\!]_K &=& \begin{cases} W & \text{if } W_p \subseteq [\![\varphi]\!]_K \\ \emptyset & \text{else} \end{cases} \quad \text{for } p \in N
\end{array}
\qquad
\begin{array}{rcl}
[\![a]\!]_K &=& \rightarrow_a \quad \text{for } a \in \mathbb{A} \\
[\![\overline{a}]\!]_K &=& \{(y,x) \mid x \rightarrow_a y\} \ \text{ for } a \in \mathbb{A} \\
[\![\varphi?]\!]_K &=& \{(w,w) \mid w \in [\![\varphi]\!]_K\} \\
[\![\pi^*]\!]_K &=& [\![\pi]\!]_K^* \\
[\![\pi_1 \text{ op } \pi_2]\!]_K &=& [\![\pi_1]\!]_K \text{ op } [\![\pi_2]\!]_K, \text{op} \in \{\cup, \cap, \circ\}
\end{array}$$

where $\mathbf{lfp}(U \mapsto [\![\varphi]\!]_{K[X \rightarrow U]})$ denotes the least fixed point of the monotone function $U \mapsto [\![\varphi]\!]_{K[X \rightarrow U]}$. A Kripke structure K is a *model* for a formula φ if for some world w of K we have $w \in [\![\varphi]\!]_K$. A formula φ is *satisfiable* if it has a model. The following two results extend corresponding theorems from [1].

Theorem 1. *Let $N \subseteq \texttt{Nom}$ be a finite set of nominals. Every satisfiable μ-ICPDL$_{\texttt{Nom}}$ formula that uses only nominals from N has a countable model of tree width at most $|N| + 2$.*

Combining this model theoretic result with automata theoretic techniques (translation into two-way alternating tree automata with a parity acceptance condition), we can show the following theorem.

Theorem 2. *Satisfiability in μ-ICPDL$_{\texttt{Nom}}$ is 2EXPTIME-complete.*

References

1. Stefan Göller, Markus Lohrey, and Casten Lutz. PDL with Intersection and Converse Is 2 EXP-Complete. In *Proceedings of the 10th International Conference on Foundations of Software Science and Computational Structures (FoSSaCS 2007)*, number 4423 in Lecture Notes in Computer Science, pages 198–212. Springer, 2007.