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Cylindric-like Algebras
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Cylindric Modal Logic

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The formalism of *cylindric modal logic* can be motivated from two directions. In its own right, it forms an interesting bridge over the gap between propositional formalisms and first-order logic, in that it formalizes first-order logic as if it were a modal formalism: The assignments of first-order variables can be seen as states or possible worlds of the modal formalism, and the quantifiers $\exists$ and $\forall$ may be studied as special cases of the modal operators $\Diamond$ and $\Box$, respectively. Elaborating this idea, we find that from this modal viewpoint, the standard semantics of first-order logic corresponds to just one of many possible classes of Kripke frames, and that other classes might be of interest as well.

From the algebraic logic perspective of this volume, cylindric modal logic provides a channel for the application of tools and ideas from modal logic in the theory of cylindric algebras. That the resulting formalism of cylindric modal logic fits in a volume on cylindric algebras, follows from the dualities between relational structures and Boolean algebras with operators. In this light, cylindric modal logic is nothing but cylindric algebra theory, studied from the dual perspective in which atom structures (of complete, atomic Boolean algebras with completely additive algebras) are the primary objects of study.

The leading question in this chapter is which facts about (representable) cylindric algebras can be explained from the general theory of modal logic and modal algebras. To mention two examples, we will see that the canonicity of the variety RCA_α follows from more general results in the duality theory of modal algebras, and that we can obtain a finite axiomatization of the equational theory of RCA_α by employing so-called non-orthodox derivation rules.

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1. First-Order Logic as Modal Logic

In this section we introduce the syntax and semantics of cylindric modal logic, we show in which sense it is a modal version of first-order logic, and we discuss some of its basic theory. Readers unfamiliar with the theory of modal logic may consult [Bla-Rij-Ven,01] for background information.

Syntax of *CML*. Starting with syntax: our purpose is to define a language that has two readings, both as (a restricted version of) first-order logic, and as a multi-modal language. Suppose that we consider a language of first-order logic with the constraints that we have a set $\{v_i \mid i < \alpha\}$ of first-order variables (with α a fixed but arbitrary ordinal), and a countable set of predicate symbols, each of arity α. The only admissible *atomic* formulas are of the form $v_i = v_j$ or $P_l(v_0 v_1 \ldots v_i \ldots)_{i<\alpha}$ (i.e., with a fixed order of the first-order variables). The motivation for adopting this particular restriction stems from the desire to stay close to the formalism of Cylindric Algebras [Hen-Mon-Tar,85, p. 152–153]; at the end of this section we will see how to handle more standard versions of first-order logic. For $\alpha < \omega$, we get a logic with finitely many variables. Such logics have been studied in the literature, for purely logical reasons [Hen,67, Tar-Giv,87, Hen-Mon-Tar,85] or because of their relation with temporal logics in computer science [Imm-Koz,87, Ott,97]; see also [And-Nem-Sai,01, Section 7] for an investigation from the algebraic logic point of view.

As their order is fixed, the variables in atomic relational formulas do not provide any information. Thus we may just as well leave them out, writing p_l for $P_l(v_0 \ldots v_i \ldots)_{i<\alpha}$, cf. [Hen-Mon-Tar,85, Remark 4.3.2]. This *restricted first-order logic* becomes *cylindric modal logic* if we replace the identity $v_i = v_j$ with the *modal constant* d_{ij}, and the existential quantification $\exists v_i$ with the *diamond* $\Diamond_i$. In order not to confuse the reader with too much notation, henceforth we will use modal notation and terminology mainly, occasionally referring to the first-order interpretation for motivation or clarification.

Definition 2.1.1. Let α be an arbitrary but fixed ordinal with $2 \leq \alpha$. CML_α is the modal language having constants d_{ij} for $i, j < \alpha$ and unary connectives ('diamonds') $\Diamond_i$ for $i < \alpha$. Given a set of propositional variables Q, the set of α-dimensional cylindric modal formulas in Q, or for short, *α-formulas (in Q)*, is built up as usual. The (modal or boolean) constants, and the variables from Q are the atomic formulas, and if φ and ψ are formulas, then so are $\varphi \wedge \psi$, $\neg\varphi$ and $\Diamond_i \varphi$. We will use standard abbreviations like $\wedge$, $\rightarrow$ and $\Box_i$.

Semantics of *CML*. Consider the basic declarative statement in first-order logic concerning the truth of a formula in a model under an assignment s:

$$\mathfrak{M} \models \varphi\ [s]. \tag{2.1.1}$$

The basic observation underlying our approach, is that we can read (2.1.1) from a modal perspective as: ‘the formula φ is true in $\mathfrak{M}$ *at state s*.’ But since we have exactly α variables at our disposal, we can identify assignments with maps from $\alpha\ \big(= \{i \mid i < \alpha\}\big)$ to U, or equivalently, with α-tuples over the domain U of the structure $\mathfrak{M}$. We will denote the set of such α-tuples by ${}^{\alpha}U$. As a consequence, we find ourselves in the setting of multi-dimensional modal logic: the universe of our modal models will be of the form ${}^{\alpha}U$ for some base set U. More information on this branch of modal logic can be found in [Kur,thisVol] of this volume; for monographs, see [Mar-Ven,97, Gab-Kur-Wol-Zak,03].

Recall the truth definition of the existential quantifier:

$$\mathfrak{M} \models \exists v_i\, \varphi[s] \text{ iff there is a } u \in U \text{ such that } \mathfrak{M} \models \varphi\big[s^i_u\big],$$

where s^i_u is the assignment defined by $s^i_u(k) = u$ if $k = i$ and $s^i_u(k) = s(k)$ otherwise. We can replace the above truth definition with the more ‘modal’ equivalent,

$$\mathfrak{M} \Vdash \Diamond_i\, \varphi[s] \text{ iff there is an assignment } s' \text{ with } s \equiv_i s' \text{ and } \mathfrak{M} \Vdash \varphi\ [s'],$$

where the binary relation $\equiv_i$ is given by

$$s \equiv_i s' \text{ iff for all } j \neq i,\ s_j = s'_j. \tag{2.1.2}$$

In other words: existential quantification behaves like a modal *diamond,* having $\equiv_i$ as its *accessibility relation.*

Since the semantics of the boolean connectives in the predicate calculus is the same as in modal logic, this shows that the inductive clauses in the truth definition of first-order logic neatly fit a modal pattern. So let us now concentrate on the atomic formulas. To start with, *equality* formulas do not cause any problem: the formula $v_i = v_j$, with truth definition

$$\mathfrak{M} \models v_i = v_j[s] \text{ iff } s \in Id_{ij}\ \big(:= \{s \in {}^{\alpha}U \mid s_i = s_j\}\big). \tag{2.1.3}$$

is indeed interpreted as a modal *constant*. Concerning the other atomic formulas, recall that in first-order logic, an α-ary predicate symbol P_l is interpreted as an α-ary relation, that is, as a subset of ${}^{\alpha}U$. This is exactly how a modal *valuation* interprets the propositional variable p_l, given our multi-dimensional setting where ${}^{\alpha}U$ provides the set of states of the model.

The above shows that, indeed, we can present the semantics of first-order logic in a completely modal framework. In order to bring this presentation in line with standard modal terminology, recall that every modal language automatically comes equipped with a relational semantics of (Kripke) *frames*, i.e. abstract structures having an (arbitrary) $n+1$-ary accessibility relation for every n-ary modal operator. From this more abstract semantic perspective on CML_{α}, its interpretation as a first-order logic can be captured by restricting the relational semantics to a rather special class of so-called *cube* frames and models.

Definition 2.1.2. An *α-frame* is a structure $\mathfrak{F} = \langle W, T_i, E_{ij} \rangle_{i,j<\alpha}$ with every $T_i \subseteq W \times W$ and every $E_{ij} \subseteq W$. The first-order language used to describe these structures (having monadic predicates E_{ij} and dyadic predicates T_i, $i, j < \alpha$), is denoted by $\mathcal{L}_{\alpha}$. Given a set U, the α-frame $\mathfrak{C}_{\alpha}(U) = \langle {}^{\alpha}U, \equiv_i, Id_{ij} \rangle_{i,j<\alpha}$, with $\equiv_i$ and Id_{ij} as given by (2.1.2) and (2.1.3), respectively, is called the *α-cube over U*. The class of α-cubes is denoted by C_{α}.

An *α-model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ with $\mathfrak{F}$ an α-frame and V a *valuation,* i.e. a map assigning a subset of the universe of $\mathfrak{F}$ to each propositional variable in the language. *Truth* of a formula φ at a world w in the model $\mathfrak{M}$ is defined by the usual induction, e.g.

$$\mathfrak{M}, w \Vdash p \iff w \in V(p),$$

$$\mathfrak{M}, w \Vdash \mathsf{d}_{ij} \iff w \in E_{ij},$$

$$\mathfrak{M}, w \Vdash \Diamond_i \psi \iff \text{there is a } v \text{ with } wT_iv \text{ and } \mathfrak{M}, v \Vdash \psi.$$

Validity of a formula or set of formulas in a model/frame/class of frames is defined and denoted as usual, e.g. $\mathsf{C}_{\alpha} \Vdash \varphi$ iff $\mathfrak{F}, V, w \Vdash \varphi$ for all frames $\mathfrak{F}$ in C_{α}, all valuations V on $\mathfrak{F}$ and all worlds w in $\mathfrak{F}$.

Relativized Cubes. In the general semantics of α-frames, states are no longer assignments but rather abstractions thereof. It is interesting to see what happens to familiar laws of the predicate calculus in this new set-up. The abstract modal perspective on the semantics of first-order logic imposes a certain *degree of validity* on familiar theorems of the predicate calculus. Some theorems are valid in all α-frames, such as distribution: $\forall v_i(\varphi \to \psi) \to (\forall v_i\varphi \to \forall v_i\psi)$. Others, such as the axiom schema $\varphi \to \exists\, v_i\varphi$ will only be valid in α-frames where T_i is a reflexive relation (below we will see many more of such correspondences). Clearly, narrowing down the class of frames increases the set of valid formulas, and vice versa.

Of particular interest are some classes of frames that differ only slightly from the cube structures, but have much nicer computational and/or logical properties. A *relativized cube* is a structure in which the states are still α-assignments on some set U (and the accessibility relations are as in $\mathfrak{C}_\alpha(U)$), but not all α-assignments on U are *available* as states. Formally, for $W \subseteq {}^\alpha U$, define $\mathfrak{C}^W_\alpha(U) := \langle W, \equiv_i \cap ({}^\alpha U \times {}^\alpha U), Id_{ij} \cap {}^\alpha U \rangle_{i,j<\alpha}$. A nice and natural intermediate class consists of multi-dimensional frames that are *locally cube*, which intuitively means that if $s \in {}^\alpha U$ is an available tuple, then any tuple 'drawing its coordinates from the set $\{s_i \mid i < \alpha\}$' should be available as well. Formally, a local cube is a relativized cube $\mathfrak{C}^W_\alpha(U)$ such that $\langle s_{\sigma(i)} \rangle_{i<\alpha} \in W$, for every $s \in W$ and every map $\sigma : \alpha \to \alpha$. Clearly, widening the semantics to such frame classes we lose some familiar first-order validities, such as $\exists\, v_i \exists\, v_j\varphi \to \exists\, v_j \exists\, v_i\varphi$; but others remain valid.

Many results are known about the CML_α-logic of these relativized cube frame classes (see [Mar-Ven,97] for an overview). We just mention Németi's seminal result [Nem,95] that the classes of relativized and of local cubes have a *decidable* logic. It was this result which led Andréka, van Benthem and Németi [And-Ben-Nem,98] to the discovery of the *guarded fragment*, an inductively defined, *decidable* fragment of first-order logic. More on this can be found in [And-Nem,thisVol] and [Ben,thisVol].

Properties of Cubes: Correspondence Theory. It is a standard modal logic question to investigate whether important frame classes admit a modal characterization, in the form of a set of modal formulas that are valid on a frame iff that frame belongs to the class. Given the fact that the cubes are not closed under taking some of the standard frame constructions such as disjoint unions or bounded morphic images, the answer for this particular class of α-frames is negative. A natural following-up problem is to study properties of the cube frames that do have a modal characterization. The next definition gathers some of these properties.

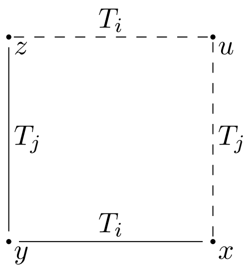
Definition 2.1.3. Consider the following pairs of cylindric modal formulas and frame formulas:

$(CM1_i)$	$p \to \Diamond_i p$	$(N1_i)$	$\forall x T_i xx$
$(CM2_i)$	$p \to \Box_i \Diamond_i p$	$(N2_i)$	$\forall xy(T_i xy \to T_i yx)$
$(CM3_i)$	$\Diamond_i \Diamond_i p \to \Diamond_i p$	$(N3_i)$	$\forall xyz\big((T_i xy \wedge T_i yz) \to T_i xz\big)$
$(CM4_{ij})$	$\Diamond_i \Diamond_j p \to \Diamond_j \Diamond_i p$	$(N4_{ij})$	$\forall xz\big(\exists y(T_i xy \wedge T_j yz) \to \exists u(T_j xu \wedge T_i uz)\big)$
$(CM5_i)$	d_{ii}	$(N5_i)$	$\forall x E_{ii} x$
$(CM6_{ij})$	$\Diamond_i(\mathsf{d}_{ij} \wedge p) \to \Box_i(\mathsf{d}_{ij} \to p)\big)$	$(N6_{ij})$	$\forall xyz\big((T_i xy \wedge E_{ij} y \wedge T_i xz \wedge E_{ij} z) \to y = z\big)$
$(CM7_{ijk})$	$\mathsf{d}_{ij} \leftrightarrow \Diamond_k(\mathsf{d}_{ik} \wedge \mathsf{d}_{kj})$	$(N7_{ijk})$	$\forall x\big(E_{ij} x \leftrightarrow \exists y(T_k xy \wedge E_{ik} y \wedge E_{kj} y)\big)$
$(CM8_{ij})$	$\big(\mathsf{d}_{ij} \wedge \Diamond_i(\neg p \wedge \Diamond_j p)\big) \to \Diamond_j(\neg \mathsf{d}_{ij} \wedge \Diamond_i p)$	$(N8_{ij})$	$\forall xz\big(E_{ij} x \wedge (\exists y T_i xy \wedge T_j yz \wedge y \neq z) \to \exists u(\neg E_{ij} u \wedge T_j xu \wedge T_i uz)\big)$

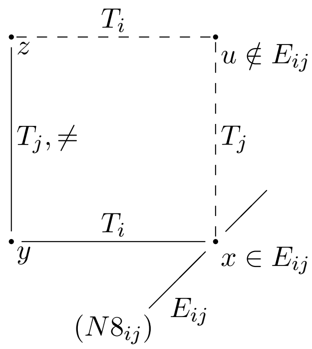
For finite α we set $(CM1) \equiv \bigwedge_i (CM1_i)$, etc., taking $(CM4) \equiv \bigwedge_{i,j}(CM4_{ij})$, $(CM6) \equiv \bigwedge_{i \neq j}(CM6_{ij})$, $(CM7) \equiv \bigwedge_{i,j,k}(CM7_{ijk})$ and $(CM8) \equiv \bigwedge_{i \neq j}(CM8_{ij})$. If $\alpha \geq \omega$, we let $(CM1), \ldots, (CM8)$ be the corresponding equation *schemata*.

An α-frame $\mathfrak{F}$ is called *cylindric* if $\mathfrak{F} \Vdash (CM1) \ldots (CM7)$, and *hypercylindric* if, in addition, $(CM8)$ is valid in it. The class of α-dimensional (hyper)cylindric frames is denoted as $(\mathsf{H})\mathsf{CF}_\alpha$.

In words, $(N1_i)$, $(N2_i)$ and $(N3_i)$ express that T_i is respectively reflexive, symmetric and transitive; together they state that T_i is an equivalence relation. $(N6_{ij})$ then means that in every T_i-equivalence class there is *at most one* element on the diagonal E_{ij} $(i \neq j)$. By $(N5_j)$ and $(N7_{jji})$ one can show that every T_i-equivalence class contains *at least one* element on the diagonal E_{ij}. Taking these observations together, we find that every world in a cylindric frame has a unique T_i-successor on the E_{ij}-diagonal. The meaning of $(N4)$ and $(N8)$ is best made clear by the following pictures:



$(N4_{ij})$



$(N8_{ij})$

The following proposition, stating that each of the above modal formulas characterizes the corresponding first-order frame condition, is a special case of a fundamental theorem in modal logic. This result, Sahlqvist's correspondence theorem, states that modal formulas of a certain syntactic shape correspond (in the sense of being equivalent as in (2.1.4) below, to a first-order formula which can be effectively computed from the modal formula.

Proposition 2.1.4. *Let* $\mathfrak{F}$ *be an* α*-frame. Then for* $l = 1, \ldots, 8$ *and* $i, j, k < \alpha$*:*

$$\mathfrak{F} \Vdash \left(CMl_{i(j(k))}\right) \iff \mathfrak{F} \models \left(Nl_{i(j(k))}\right). \tag{2.1.4}$$

Using this result, together with [Hen-Mon-Tar,85, Theorem 2.7.40] it is not hard to prove the following proposition which explains our terminology.

Proposition 2.1.5. *An* α*-frame* $\mathfrak{F}$ *is a cylindric frame iff* $\mathfrak{F}^+$ *is a cylindric algebra.*

Modalizing Standard First-Order Logic. At this point the reader may complain that the formalism that we have been 'modalizing' is not first-order logic at all, but at best a rather peculiar variant of it. So, to justify this section's title, let us briefly see which adaptations to make, in order to cover more standard versions of the predicate calculus.

To start with, the fact that we consider versions of first-order logic with *more than* ω many variables is not a problem at all. In fact, our only reason for allowing these was to be able to cover cylindric algebras of arbitrary dimension. We may just as well confine attention to the case $\alpha \leq \omega$.

Second, while we saw that all inductive clauses in the semantics of first-order logic are in complete accordance with the modal pattern, and that the identity formulas can be treated as modal constants, the other atomic formulas are more problematic. First of all, predicate symbols in first-order logic may have an arbitrary arity, whereas our propositional variables are interpreted as subsets of ${}^{\alpha}U$ for a fixed α. This imbalance could be corrected semantically by restricting the valuation of a proposition letter p corresponding to a k-ary predicate P, to those subsets of ${}^{\alpha}U$ that are closed under the relation $\equiv_i$ for $i \geq k$, or syntactically (in the case $\alpha = n < \omega$ and $k < n$) by translating the atomic formula $Pv_0 \cdots v_{k-1}$ as $\Diamond_k \cdots \Diamond_{n-1} p$. Another problem, however, is that, even if we restrict to the n-variable fragment of first-order logic with n-ary predicate symbols, our modal formalism can only deal with a restricted version of first-order logic. Because our valuations on n-cubes are in one-one correspondence with the interpretations of n-ary predicates, we obtain that $\mathfrak{M}, s \Vdash p$ (in the modal sense) iff $\mathfrak{M} \models Pv_0 \cdots v_{n-1}$ (in the first-order sense). But how to handle atomic formulas where the first-order variables do not occur in the fixed order $v_0 \cdots v_{n-1}$, that is, formulas of the form $Pv_{\sigma(0)} \cdots v_{\sigma(n-1)}$, for some map $\sigma : n \to n$? Atomic formulas with some *multiple* occurrence of a variable can be rewritten as non-atomic formulas involving only unproblematic atomic formulas, see [Hen-Mon-Tar,85, p. 152] (for instance: $Pv_0v_0 \iff \exists v_1(v_0 = v_1 \wedge Pv_0v_1)$.

This leaves the case what to do with atoms of the form $Pv_{\sigma(0)} \cdots v_{\sigma(n)}$ with $\sigma : n \to n$ a *permutation.* The crucial observation is that for any permutation $\sigma \in {}^{n}n$, we have

$$\mathfrak{M} \models Pv_{\sigma(0)} \cdots v_{\sigma(n-1)}[s] \iff \mathfrak{M} \models Pv_0 \cdots v_{n-1}[s \circ \sigma],$$

where $s \circ \sigma$ is the composition of $\sigma : n \to n$ and $s : n \to U$. So if we add an explicit *substitution operators* $\odot_\sigma$ to the language, with semantics

$$\mathfrak{M}, s \Vdash \odot_\sigma \varphi \iff \mathfrak{M}, s \circ \sigma \Vdash \varphi,$$

we can indeed handle any atomic formula of the form $Pv_{\sigma(0)} \cdots v_{\sigma(n-1)}$, translating it as $\odot_\sigma p$. The resulting system, which can be seen as the modal version of polyadic equality algebras, has been developed along the same lines as cylindric modal logic, see Venema [Ven,95a] for details.

Type-Free and Schema Validity. The ω-dimensional version of our logic may not correspond directly to the standard predicate calculus, it does encode two rather interesting notions related to first-order logic: *type-free validity*, introduced in [Hen-Mon-Tar,85, 4.3.65] (see also Simon [Sim,91]) and *scheme validity,* see Németi [Nem,87a].

Type-free logic arises if we think of CML_ω-formulas as so-called *type-free* formulas. A type is a map $\rho : Q \to \omega$ assigning a finite arity to each propositional variable/predicate letter in Q, and the ρ-instantiation φ^ρ of a formula $\varphi \in CML_\omega$ is simply the first-order formula we obtain from φ by simultaneously replacing every propositional variable p_l occurring in φ with the formula $P_l v_0 \cdots v_{\rho(p_l)-1}$ (and reading the identity constants and modal operators in the first-order way). A CML_ω-formula is *type-free valid* if each of its typed instances is valid (as a first-order formula).

Similarly, CML_ω-formulas can be seen as first-order *schemas:* formally we may define a *first-order instance* of φ as the result of uniformly substituting arbitrary first-order formulas for the propositional variables in φ. We call φ *schema valid* if each of these first-order instances is valid as a first-order formula.

It is not hard to show that these two notions are equivalent, and closely related to the cube semantics of CML_ω. More precisely, for each CML_ω-formula φ we have that

(2.1.5) $$\mathsf{C}_\omega \Vdash \varphi \iff \varphi \text{ is type-free valid } \iff \varphi \text{ is schema valid.}$$

2. Cylindric Modal Logic and Cylindric Algebras

In this section we discuss how cylindric modal logic fits in the theory of cylindric algebras. Roughly speaking, in cylindric modal logic one focuses on the *atom structures* associated with cylindric algebras. Conversely, from the perspective of modal logic, cylindric algebras provide an interesting class of *modal algebras.* Many of the notions defined here are discussed in [Hen-Mon-Tar,85, Section 2.7]; see also [Kur,thisVol] and [Bez,thisVol]. For an overview of the algebraic approach towards modal logic the reader may consult [Ven,07].

Definition 2.2.1. A cylindric-type modal algebra of dimension α, or shortly: an *α-modal algebra* is an algebra $\mathbb{A} = \langle A, +, \cdot, -, 0, 1, \Diamond_i, \mathsf{d}_{ij}\rangle_{i,j<\alpha}$ of type CML_α, where $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra, and each $\Diamond_i$ is a unary operator on $\langle A, +, \cdot, -, 0, 1\rangle$, that is, a normal ($\Diamond_i 0 = 0$) and additive ($\Diamond_i(x + y) = \Diamond_i x + \Diamond_i y$) operation.

Frames and Algebras. At the basis of the algebraic perspective on modal logic lies the following construction of an algebra from a frame.

Definition 2.2.2. The *complex algebra* of an α-frame $\mathfrak{F} = \langle W, T_i, E_{ij}\rangle_{i,j<\alpha}$ is the structure $\mathfrak{F}^+ := \left\langle \mathcal{P}(W), \cup, \cap, \sim_W, \varnothing, W, \langle T_i\rangle, E_{ij}\right\rangle$, where the map $\langle T_i\rangle : \mathcal{P}(W) \to \mathcal{P}(W)$ is given by

$$\langle T_i\rangle(X) := \{w \in W \mid T_i wx \text{ for some } x \in X\}.$$

Given a class K of frames, we let $\mathsf{Cm\,K}$ denote the associated class of complex algebras.

Intuitively, complex algebras are the algebraic encodings of frames, and are often thought of as 'concrete' α-modal algebras, in the same sense that power-set algebras are concrete Boolean algebras. An indication of their fundamental importance, and a first link between the areas of cylindric modal logic and that of cylindric algebras, is the following observation.

Theorem 2.2.3. RCA_α *is the variety generated by* $\mathsf{Cm\,C}_\alpha$.

Proof. Immediate by the observation that RCA_α is the variety generated by the class of α-dimensional full cylindric set algebras, which is precisely the class $\mathsf{Cm\,C}_\alpha$ of complex algebras of cubes. ■

The original frame can be retrieved as the *atom structure* of its complex algebra, where the atom structure of an atomic α-modal algebra $\mathbb{A}$ is defined as a certain α-frame based on the collection of atoms of the (Boolean reduct of) $\mathbb{A}$. The operations of taking complex algebras and atom structures form part of a categorical duality, but we lack the space for going into detail here.

From an arbitrary (that is, not necessarily atomic) α-modal algebra we can obtain an α-frame as follows.

Definition 2.2.4. The *ultrafilter frame* of an α-modal algebra $\mathfrak{A} = \left\langle A, +, \cdot, -, 0, 1, \Diamond_i, \mathsf{d}_{ij}\right\rangle_{i,j<\alpha}$ is the α-frame $\mathfrak{A}_\bullet := \langle \mathsf{Uf}\,\mathfrak{A}, \mathsf{S}_{\Diamond_i}, \mathsf{D}_{ij}\rangle_{i,j<\alpha}$, where $\mathsf{Uf}\,\mathfrak{A}$ denotes the set of ultrafilters of the (Boolean reduct of) $\mathfrak{A}$, $\mathsf{D}_{ij} := \{u \in \mathsf{Uf}\,\mathfrak{A} \mid \mathsf{d}_{ij} \in u\}$, and $\mathsf{S}_{\Diamond_i} \subseteq \mathsf{Uf}\,\mathfrak{A} \times \mathsf{Uf}\,\mathfrak{A}$ is given by $S_{\Diamond_i}uv$ iff $\Diamond_i a \in u$ for all $a \in v$.

The operations of taking complex algebras and ultrafilter frames can be extended to functors between the categories of α-frames (with so-called bounded morphisms as arrows), and α-modal algebras (with homomorphisms). They do not provide a full-blown categorical duality, however, unless we restrict both categories to the full subcategories of their finite members. Nevertheless, the composition of the operations $(\cdot)^+$ and $(\cdot)_\bullet$ provides one of the key notions in the area.

Definition 2.2.5. Given an α-modal algebra $\mathfrak{A}$, the algebra $(\mathfrak{A}_\bullet)^+$ is called the perfect or *canonical extension* of $\mathfrak{A}$. A class K of α-modal algebras is *canonical* if it is closed under taking canonical extensions.

As an extension of Stone's representation theorem for Boolean algebras, Jónsson and Tarski [Jon-Tar,51] proved that every α-modal algebra can be embedded in its perfect extension. This shows that every α-modal algebra $\mathfrak{A}$ can be *represented* as a subalgebra of a concrete algebra, namely, the complex algebra $(\mathfrak{A}_\bullet)^+$ of the frame $\mathfrak{A}_\bullet$. As a consequence, canonicity is a very desirable property for a variety (or class) to have because it means that every algebra in the variety can be represented as a concrete algebra *in the same variety.*

Theorem 2.2.6. *The variety* RCA_α *is canonical.*

This result, stated as Theorem 2.7.24(ii) in [Hen-Mon-Tar,85], can be seen as an instantiation of a more general result in modal logic by Fine which states that every *elementary* (first-order definable) frame class generates a canonical variety. Indeed, it is not hard to come up with a set of first-order sentences that characterize the class of α-cubes. Theorem 2.2.6 and variants are investigated from this perspective in [And-Gol-Nem,98].

Formulas, Terms and Equations. Generally, the starting point in algebraic logic is the formal identification of logical connectives with algebraic function symbols, and, consequently, of formulas with algebraic terms. Given a sufficiently expressive repertoire of connectives, the link between logic and algebra can be extended to the semantics of formulas and equations, respectively. For instance, modulo some simple translations between formulas and equations, the notion of modal validity on a frame coincides with that of equational validity on its complex algebra:

$$\mathfrak{F} \Vdash \varphi \quad \text{iff} \quad \mathfrak{F}^+ \models \varphi \approx \top, \tag{2.2.1}$$

$$\mathfrak{F} \Vdash \varphi \leftrightarrow \psi \quad \text{iff} \quad \mathfrak{F}^+ \models \varphi \approx \psi. \tag{2.2.2}$$

On the basis of (2.2.1) we will say that an algebra validates a modal formula φ if it validates its equational translation $\varphi \approx \top$.

LOGICS AND VARIETIES. Finally, we turn to axiomatization issues. In modal logic, the key concept is that of a *normal modal logic.*

Definition 2.2.7. A *normal cylindric modal logic of dimension* α, or an α-logic, is a set of CML_α-formulas containing

(CT)	all propositional tautologies
$(DB_{\Box_i})$	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$

which is closed under the derivation rules, Modus Ponens, Universal Generalization and Substitution:

(MP)	$\vdash \varphi, \vdash \varphi \to \psi \;\Rightarrow\; \psi$,
$(UG_{\Box_i})$	$\vdash \varphi \;\Rightarrow \vdash \Box_i \varphi$,
(SUB)	$\vdash \varphi \;\Rightarrow \vdash \sigma(\varphi)$, for any substitution σ of formulas for propositional variables in φ.

Given a set Γ of formulas, we let $\mathsf{K}_\alpha.\Gamma$ denote the normal cylindric modal logic *axiomatized by* Γ, that is, the smallest α-logic containing Γ, and write K_α for $\mathsf{K}_\alpha.\varnothing$.

Algebraically, normal modal logics correspond to deductively closed sets of equations, and hence, by Birkhoff's completeness for equational logic, also to varieties of modal algebras. Given a set Γ of formulas, let V_Γ denote the variety of α-modal algebras axiomatized by the set $\{\gamma \approx \top \mid \gamma \in \Gamma\}$ of equations.

Proposition 2.2.8. *The map $L \mapsto \mathsf{V}_L$ is a dual isomorphism between the lattice of normal cylindric modal logics of dimension α, and the lattice of varieties of α-modal algebras. The variety V_L algebraizes the logic L, in the sense that*

$$\varphi \in L \quad \textit{iff} \quad \mathsf{V}_L \models \varphi \approx \top, \tag{2.2.3}$$

$$\varphi \leftrightarrow \psi \in L \quad \textit{iff} \quad \mathsf{V}_L \models \varphi \approx \psi. \tag{2.2.4}$$

Note that the algebraization given by the equivalences (2.2.3) and (2.2.4) is of a very simple nature. Generally, the notion of a class of algebras *algebraizing* a logic [Blo-Pig,89], is more sophisticated, dealing with logics as consequence relations rather than as sets of theorems, and admitting more complex translations between formulas and equations than the ones above.

Axiomatizing the Cubes. Given Theorem 2.2.3 and the connections (2.2.1) and (2.2.2), axiomatizing the modal logic of the class of α-cubes, and axiomatizing the equational theory of variety of representable cylindric algebras amounts to the same thing. Unfortunately, for $\alpha > 2$, it follows from the nonfinite axiomatizability results of Monk, Andréka, and others, that if we only allow orthodox derivation systems then there is no finite set of axioms and rules that, when added to K_α, yields a complete axiomatization for the class of cubes. Nevertheless, we can see how far the modal versions of the cylindric algebra axioms bring us.

Definition 2.2.9. Let $\mathbf{CML}_\alpha$ and $\mathbf{HCML}_\alpha$ be the normal modal logics axiomatized by the formulas (CM1–7) and (CM1–8), respectively.

The following result of Venema [Ven,95b] is another immediate consequence of the Sahlqvist shape of the formulas (CM1–8).

Theorem 2.2.10. *The systems* $\mathbf{CML}_\alpha$ *and* $\mathbf{HCML}_\alpha$ *are sound and complete for the classes* CF_α *and* HCF_α*, respectively.*

Dimension Two. In the case $\alpha = 2$, however, the system $\mathbf{HCML}_2$ is sound and complete for the class of squares (which is our usual term to refer to the cubes of dimension two). Consequently, the set of equations corresponding to (CM1–8) provides a finite axiomatization of the class RCA_2. Thus our axiom (CM8) provides an alternative to Henkin's equation, see [Hen-Mon-Tar,85, Theorem 3.2.65(ii)], or [Rij-Ven,95] for a more detailed discussion. We refer to [Bez,thisVol] for more information on the two-dimensional case.

3. Completeness for Cylindric Modal Logic

Introduction. As an application of modal logic in the theory of cylindric algebras, in this section we will see that if one is willing to generalize the definition of an axiomatization by admitting so-called *unorthodox* derivation rules, then the finite axiomatization problem can be overcome in a fairly simple and elegant way. Here we call a rule unorthodox if it has a side condition to the effect that the applicable instances of the rule are not closed under taking substitutions – in other words, unorthodox rules are not structural in the sense of Blok and Pigozzi [Blo-Pig,89].

Obviously, with this definition many different kinds of rules may be classified as being unorthodox – the ones we employ here are characterized by a side condition involving the occurrences of variables in the premisse of the rule. The use of such rules in modal logic goes back to the work of Gabbay [Gab,81] and Burgess [Bur,80]. More specifically, here we will focus on a method of proving completeness via a rule that involves the so-called *difference operator* and that shares with Sahlqvist's theorem the feature of automatically turning characterizations into axiomatizations; see [Ven,93] for a detailed discussion.

Before continuing, let us be a bit more precise about the distinction between logics and derivation systems. We shall call a derivation system a pair consisting of a set of formulas (called axioms), and a set of derivation rules. A *derivation* in such a system is defined as a nonempty, finite sequence $\varphi_0, \ldots, \varphi_n$ such that every φ_i is either an axiom or obtainable from $\varphi_0, \ldots, \varphi_{i-1}$ by a derivation rule. For instance, we think of the normal modal logic $\mathsf{K}_\alpha.\Gamma$ as the derivation system with as its axioms, besides Γ, all classical tautologies and the modal distribution axioms, and as its derivation rules: modus ponens, universal generalization, and substitution. A *theorem* of a derivation system is any formula that can appear as the last item of a derivation. Hence, any derivation system containing the above-mentioned triple as derivation rules, will produce, as its set of theorems, a normal modal logic. Theoremhood of a formula φ in the system $\mathbf{HCML}_\alpha$ is denoted by $\vdash_\alpha \varphi$.

Characterizing the n-Cubes. Our unorthodox completeness theorem will be based on a rather special characterization of the n-cubes – where for the moment we fix a finite ordinal n with $2 \leq n$. The starting point for this characterization is the observation that the *inequality relation* $\neq$ on a cube can be obtained in a nice, ‘modal’ way, namely, as a certain composition of the cube's accessibility relations. As a corollay, we find that on the class of n-cubes, the difference operator is term definable as a compound modality. (A compound modality is a modal formula $\varphi(p)$ with one free variable p that is composed from p using diamonds, disjunctions, and conjunctions with variable-free formulas.)

The modal perspective on the inequality relation is based on two simple facts about cubes. First, two tuples are distinct iff they differ in at least one coordinate. And second, two tuples, say u and v, differ in some coordinate, say, 0, iff one can make the following ‘modal walk’ from $u = (u_0, \dots, u_{n-1})$ to $v = (v_0, \dots, v_{n-1})$ along the accessibility relations: (i) first move along T_1 to the diagonal E_{01}, arriving at $(u_0, u_0, u_2, \dots, u_{n-1})$; (ii) continue by moving along T_0 *off* the diagonal, arriving at $(v_0, u_0, u_2, \dots, u_{n-1})$; (iii) finally, move along $T_1, \dots, T_{n-1}$, arriving at v. Turning to the general setting of hypercylindric n-frames, this ‘modal walk’ can be formalized to the following definition of a binary relation R^n.

Definition 2.3.1. For an arbitrary hypercylindric n-frame $\mathfrak{F} = \big(W, T_i, E_{ij}\big)_{i,j<n}$, define $f_{ij}(u)$, H_i^n, H^n and R^n as follows: $f_{ij}(u)$ is the unique v such that $T_i uv$ and $E_{ij} v$. H^n (resp. H_i^n, $i < \alpha$) is the composition of all the T-relations, resp. all the T-relations minus T_i, i.e.

$$H^n = T_0 \circ T_1 \circ \dots \circ T_{n-1},$$

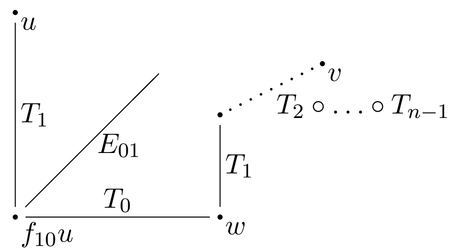
$$H_i^n = T_0 \circ T_1 \circ \dots \circ T_{i-1} \circ T_{i+1} \circ \dots \circ T_{n-1}.$$

For a world u in $\mathfrak{F}$, the set $H_i^n(u) = \{v \mid \mathfrak{F} \models H_i^n uv\}$ is called the i-hyperplane through u. R^n is given by

(2.3.1)
$$R^n = \Big\{(u,v) \in W \times W \mid \mathfrak{F} \models \bigvee_i \bigvee_{j \neq i} \exists w \big(T_i f_{ji} uw \wedge \neg E_{ij} w \wedge H_i^n wv\big)\Big\}.$$

It is easily verified that in a *cube* $\mathbb{C}_n(U)$, we have that $H^n = {}^nU \times {}^nU$, that $H_i^n = \big\{(u,v) \in {}^nU \mid u_i = v_i\big\}$, and that the function f_{ij} is given by (taking $i = 1$ and $j = 0$) $f_{10}(u_0, \dots, u_{n-1}) = (u_0, u_0, u_2, \dots, u_{n-1})$. In a hypercylindric frame, the function f_{ij} is the projection along T_i on the E_{ij} diagonal. H^n and each H_i^n is an equivalence relation, and it does not matter in which order we compose the T_j relations to define them. The i-hyperplanes are the equivalence classes of H_i^n.

In order to understand the definition of R^n, consider the following figure depicting the case where $R^n uv$ because $T_0 f_{10} uw \wedge \neg E_{01} w \wedge H_0^n wv$ (that is, take $i = 0$ and $j = 0$ in (2.3.1)):



Here v lies in the hyperplane through w and 'orthogonal' to the 'line' T_0. Then, given the description, just above Definition 2.3.1, of the 'modal walk' from a tuple u to an arbitrary tuple v with $u_0 \neq v_0$, it is not hard to see that on n-cubes, R^n is the *inequality* relation. The point of our characterization is that the latter property exactly singles out the cubes among the hypercylindric frames:

Theorem 2.3.2. *Let $\mathfrak{F} = \langle W, T_i, E_{ij} \rangle_{i,j<n}$ be an n-frame. Then $\mathfrak{F}$ is isomorphic to a cube iff it is hypercylindric and R^n is the inequality relation on W.*

For a *proof* of this Theorem, which technically can be see as the frame version of Theorem 3.2.5 in [Hen-Mon-Tar,85], the reader is referred to [Ven,95b]. Concerning the result itself, one interesting observation is that it gives a *first-order* characterization of the cubes. In Theorem 2.2.6 we already saw an immediate corollary of this: the variety of representable cylindric algebras is canonical. However, as a first-order characterization, Theorem 2.3.2 is rather involved; there are certainly simpler alternatives available.

The interest in the specific form of this characterization lies in the fact that on the class of hypercylindric frames, R^n is the accessibility relation of the following compound modality.

Definition 2.3.3. Define the following abbreviated operator D_n:

$$D^n\varphi := \bigvee_i \bigvee_{j \neq i} \Diamond_j \big(\mathsf{d}_{ij} \wedge \Diamond_i (-\mathsf{d}_{ij} \wedge \Diamond_0 \ldots \Diamond_{i-1} \Diamond_{i+1} \ldots \Diamond_{n-1} \varphi) \big),$$

Note that D_n is defined to make the relation R^n act as its accessibility relation, i.e. for any hypercylindric n-model we have

$$\mathfrak{M}, u \models D_n\varphi \iff \text{there is a } v \text{ with } R^n uv \text{ and } \mathfrak{M}, v \models \varphi.$$

But then Theorem 2.3.2 states that the cubes are exactly the class of hypercylindric frames on which D_n acts as the so-called *difference operator.*

DIFFERENCE OPERATOR. The difference operator is a modality D that has the *inequality relation* $\neq$ as its intended accessibility relation:

$$\mathfrak{M}, s \models D\varphi \text{ iff } \mathfrak{M}, t \models \varphi \text{ for some } t \neq s.$$

This operator increases the expressivity of modal languages; it can be used for instance to express frame properties that are otherwise inexpressible by modal formulas, such as irreflexivity ($\Diamond p \to Dp$) or antisymmetry ($p \wedge \Diamond(\neg p \wedge \Diamond p) \to Dp$). An interesting aspect of the difference operator is that it allows one to *name* states. Consider the formula

$$\mathtt{name}_D(\varphi) := \varphi \wedge \neg D\varphi,$$

then clearly the formula $\mathtt{name}_D(\varphi)$ holds at a state s iff s is the *only* state where φ is true. For more background on the difference operator, the reader is referred to [Rij,92, Ven,93].

Now suppose that we want to *axiomatize* the behaviour of the difference modality, in a setting where D is an additional modality added to the language of, say, cylindric modal logic. Many properties of the inequality relation are easy to characterize and axiomatize: symmetry by the modal formula (D1) $p \to \neg D \neg Dp$, pseudo-transitivity ($\forall xyz\,(Rxy \wedge Ryz \to Rxz \vee x = z)$) by (D2) $DDp \to p \vee Dp$, and the inclusion property ($\forall xy\, R_i xy \to x = y \vee Rxy$) by (D3) $\Diamond_i p \to p \vee Dp$, for all diamonds $\Diamond_i$. The problem is the property of *irreflexivity* which cannot be characterized by a formula, at least not in the standard way. Consider, on the other hand, the following *derivation rule:*

$$(IR_D) \qquad\qquad \vdash \mathtt{name}_D(p) \to \varphi \;\Rightarrow \vdash \varphi, \quad \text{if} \quad p \notin \varphi.$$

It should be obvious where this rule fails to be orthodox: any renaming which replaces a proposition letter in φ with p may transform an applicable instance of (IR_D) into a forbidden one. In order to get an understanding of what this rule does, let us first check its *soundness.* Using contraposition, assume that $\not\models \varphi$. That is, there is some model $\mathfrak{M} = (\mathfrak{F}, V)$ and a state s such that $\mathfrak{M}, s \not\models \varphi$. Now modify the valuation V to a valuation V' by changing the interpretation of p to the singleton $\{s\}$. Then $\mathtt{name}_D(p)$ is true at s by definition, whereas the formula φ remains false since its truth is not affected by changing the meaning of p. As a consequence we see that $\not\models \mathtt{name}_D(p) \to \varphi$, as required.

Often we are dealing with a situation where we do not have a *primitive* modality D, but rather some compound modality D_c which behaves like the difference operator on the intended class of frames.

Definition 2.3.4. Let D_c be some compound modality. Given a derivation system $\mathbf{\Delta}$, we define $\mathbf{\Delta}.D_c^+$ as its extension obtained by adding the D_c-versions of the axioms D(1–3) and the irreflexivity rule (IR_D).

Theorem 2.3.5. *Let S be a set of Sahlqvist axioms containing the axiom $(CM2)$ for each modality, and let D_{K} be a compound modality such that K is the class of n-frames that validate S and on which D_{K} acts as the difference operator. Then $\mathsf{K}.S.D_{\mathsf{K}}^+$ is sound and complete for the class K.*

Proof Sketch. To prove *soundness* is left as an exercise for the reader. The *completeness* proof falls out into three parts. First, using a multi-dimensional Lindenbaum Lemma, one may show that for every consistent formula ξ there is a set W^ξ of maximal consistent sets (MCSs) with the Properties 1–3 below (here the relation R_i between MCSs is defined as usual: $R_i\Gamma\Delta$ iff $\Diamond_i\varphi \in \Gamma$ for every $\varphi \in \Delta$):

1. There is an MCS $\Xi \in W^\xi$ containing ξ;
2. W^ξ satisfies an *Existence Lemma:* for every formula φ and for every $\Gamma \in W^\xi$ we have
$$\Diamond_i\varphi \in \Gamma \;\text{ iff }\; \varphi \in \Delta \;\text{ for some }\; \Delta \in W^\xi \;\text{ with }\; \Gamma R_i \Delta;$$
3. Every MCS has a name: for every Γ there is a proposition letter p^Γ such that
$$\mathtt{name}_{D_{\mathsf{K}}}\left(p^\Gamma\right) \in \Theta \;\text{ iff }\; \Theta = \Gamma.$$

Second, on the basis of these properties it makes sense to define the following variants of the canonical frame and model. $\mathfrak{F}^\xi$ is the structure $\mathfrak{F}^\xi = \left\langle W^\xi, R_i, \mathsf{D}_{ij}\right\rangle_{i,j<\alpha}$, with $\mathsf{D}_{ij} := \{\Gamma \in W^\xi \mid \mathsf{d}_{ij} \in \Gamma\}$, and $\mathfrak{M}^\xi$ is the model on $\mathfrak{F}^\xi$ determined by the valuation V given by $V(p) := \{\Gamma \in W^\xi \mid p \in \Gamma\}$. Without loss of generality we may assume that the frame $\mathfrak{F}^\xi$ is point-generated from Ξ, that is, to every $\Gamma \in W^\xi$ there is a path from Ξ following the relations R_i. The motivation behind these definitions is that they enable us to prove (by a straightforward formula induction) the following *Truth Lemma:*

$$\mathfrak{M}^{\xi}, \Gamma \models \varphi \;\text{ iff }\; \varphi \in \Gamma, \tag{2.3.2}$$

for every MCS Γ and formula φ. And as an immediate corollary of the truth lemma and property (1) above it follows that the formula ξ is satisfied in the model $\mathfrak{M}^{\xi}$.

It remains to show that we have satisfied ξ in a model of the right *kind,* that is, we have to prove that the underlying frame $\mathfrak{F}^{\xi}$ belongs to the class K. By our assumption on K, this boils down to showing that (i) the axioms from S are valid on $\mathfrak{F}$, and (ii) that D_{K} acts as the difference operator on $\mathfrak{F}$.

For (i) we just consider a representative example: the formula $(CM4_{01})$: $\Diamond_0\Diamond_1 p \to \Diamond_1\Diamond_0 p$. By Sahlqvist correspondence (Lemma 2.1.4 above) it suffices to prove that $\mathfrak{F} \models (N4_{01})$. Assume that $\Gamma R_0 \Delta R_1 \Theta$, then by the properties established above, we find $\Diamond_0\Diamond_1\,\mathtt{name}_{D_{\mathsf{K}}}(p^{\Theta}) \in \Gamma$, and so by maximal consistency, also $\Diamond_1\Diamond_0\,\mathtt{name}_{D_{\mathsf{K}}}(p^{\Theta}) \in \Gamma$. This gives MCSs Π and Θ' such that $\Gamma R_1 \Pi R_0 \Theta'$ and $\mathtt{name}_{D_{\mathsf{K}}}(p^{\Theta}) \in \Theta'$. It follows that $\Theta = \Theta'$, and so Π is the required MCS such that $\Gamma R_1 \Pi$ and $\Pi R_0 \Theta$.

In order to prove (ii) one may show that the relation $R_D := \big\{(\Gamma, \Delta) \in W^{\xi} \times W^{\xi} \mid D\varphi \in \Gamma \text{ for all } \varphi \in \Delta\big\}$ is the inequality relation on $\mathfrak{F}^{\xi}$. For the inclusion $R_D \subseteq\, \neq$ one needs property 3 above, while for the opposite inclusion we use the D_c-axioms, together with the fact that $\mathfrak{F}^{\xi}$ is point-generated from Ξ, to show that any pair of MCSs in W^{ξ} is linked by the relation $R_D \cup =$. Further details are left to the reader. ■

Axiomatizing the n-Cubes. The completeness result for the n-cubes is now a fairly straightforward consequence of earlier results.

Definition 2.3.6. For finite dimensions n, $\mathbf{HCML}_n^+$ is the derivation system $\mathbf{HCML}_n$ extended with the *Irreflexivity Rule for* D_n:

$$\vdash \mathtt{name}_{D_n}(p) \to \varphi \;\Rightarrow \vdash \varphi, \quad \text{if } \; p \notin \varphi. \tag{IR_{D_n}}$$

Define $\mathbf{HCML}_\omega^+$ as the system $\mathbf{HCML}_\omega$ extended with the schema of rules $\{IR_{D_n} \mid n < \omega\}$. For $\alpha > \omega$ we add besides this set, the following schema:

$$\{\vdash \varphi \;\Rightarrow \vdash \varphi^{\tau} \mid \tau : \alpha \mapsto \alpha \text{ is a bijection}\},$$

where φ^{τ} is the formula one obtains from φ by substituting $\Diamond_{\tau(i)}$ and $\mathsf{d}_{\tau(i)\tau(j)}$ for every occurrence of $\Diamond_i$ resp d_{ij}.

The following result is due to Venema [Ven,95b].

Theorem 2.3.7. *For any ordinal α, the derivation system $\mathbf{HCML}^+_\alpha$ is a sound and complete axiomatization for the class of α-dimensional cubes. That is, for all CML_α-formulas φ:*

$$\vdash^+_\alpha \varphi \text{ iff } \mathsf{C}_\alpha \models \varphi.$$

Proof. Again we leave it for the reader to prove soundness. For completeness, the case for finite α is a more or less straightforward corollary to the Theorems 2.3.2 and 2.3.5. (We omit the rather technical details of the proof that the D_n-versions of the D-axioms are derivable in the logic $\mathbf{HCML}_n$.)

When it comes to infinite dimensions we confine ourselves to the case $\alpha = \omega$. Let φ be an ω-formula such that $\mathsf{C}_\omega \models \varphi$. As there are only finitely many symbols occurring in φ, there is an $n < \omega$ such that φ is an n-formula. A relatively simple argument shows that for all ordinals $\beta < \gamma$, and all β-formulas ψ : $\mathsf{C}_\beta \models \psi$ iff $\mathsf{C}_\gamma \models \psi$. From this we conclude that our φ is valid in C_n, so that by finite-dimensional completeness we obtain $\vdash^+_n \varphi$. Now $\vdash^+_\omega \varphi$ follows as $\mathbf{HCML}^+_\omega$ is an extension of $\mathbf{HCML}^+_n$. ■

Given the connections between normal cylindric modal logics and equational theories, and the equivalence (2.1.5) of cube validity, type-free validity, and schema validity for CML_α-formulas, it is straightforward to verify that Theorem 2.3.7 also provides finite, complete axiomatizations for the equational theory of RCA_α, and for both type-free and first-order schema validity. The system $\mathbf{HCML}^+_\omega$ thus indicates a positive solution to Problem 4.16 of [Hen-Mon-Tar,85].

From the perspective of cylindric algebras, what goes on here can be reformulated as follows. Let HCA_n be the class of n-dimensional cylindric algebras satisfying the additional equation corresponding to the axiom ($CM8$), and consider Theorem 3.2.5 from [Hen-Mon-Tar,85] which states that an algebra in HCA_n is representable if it is *rich,* that is, it has sufficiently many elements that are satisfy a certain equation. The point is that this richness condition may be transformed into a *derivation rule* that is non-orthodox in the sense discussed at the beginning of this section. Nevertheless, this rule is sound, and when we add it to the equational axiomatization for HCA_n we obtain a finite, complete axiomatization for the variety of representable cylindric algebras.

Mutatis mutandis, this approach works in other situations as well. For instance, in [Ven,98] the author obtained a finite axiomatization for the class of representable diagonal-free algebras, using a nonorthodox derivation rule inspired by a representation result for so-called *rectangularly dense* algebras. (This and related notions of density, including the above-mentioned concept of 'richness', are discussed in detail in [And-Giv-Mik-Nem-Sim,98].) A different type of rule was used by Simon [Sim,91] to obtain a complete axiomatization for the type-free valid formulas (and hence, for the equational theory of RCA_ω).