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A strict implication calculus for compact Hausdorff spaces

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We introduce a simple modal calculus for compact Hausdorff spaces. The language of our system extends that of propositional logic with a strict implication connective, which, as shown in earlier work, algebraically corresponds to the notion of a subordination on Boolean algebras. Our base system is a strict implication calculus SIC, to which we associate a variety SIA of strict implication algebras. We also study the symmetric strict implication calculus $\mathsf{S^2IC}$, which is an extension of SIC, and prove that $\mathsf{S^2IC}$ is strongly sound and complete with respect to de Vries algebras. By de Vries duality, this yields completeness of $\mathsf{S^2IC}$ with respect to compact Hausdorff spaces. Since some of the defining axioms of de Vries algebras are Π_2-sentences, we develop the corresponding theory of non-standard rules, which we term Π_2-rules. We study the resulting inductive elementary classes of algebras, and give a general criterion of admissibility for Π_2-rules. We also compare our approach to approaches in the literature that are related to our work.[1]

1. Introduction

Extending Stone's seminal representation theorems for Boolean algebras [25] and distributive lattices [26], categorical dualities linking algebra and topology have been of fundamental importance in the development of the 20th century mathematics in general [22], and of logic and theoretical computer science in particular [18]. With algebras corresponding to the syntactic, deductive side of logical systems, and topological spaces

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[1] The results presented in this paper were first reported in [24] (see also [5]).

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to their semantics, Stone-type duality theory provides an elegant and useful mathematical framework for studying various properties of logical systems. In many particular cases, one sees natural specimens of logics, classes of algebras, and classes of topologies coming together. Out of a multitude of examples of such triples, we mention: (i) classical logic/Boolean algebras/Stone spaces [2]; (ii) intuitionistic logic/Heyting algebras/Esakia spaces [15]; and (iii) modal logic/modal algebras/topological Kripke frames [7].

Our aim is to add to the study of these 'logic/algebra/topology' triples by providing a simple logical calculus for reasoning about compact Hausdorff spaces—a widely studied class of spaces, properly containing the class of Stone spaces. We do this by generalizing the classical setting. Namely, we extend the classical propositional language with a new logical connective of *strict implication*, which admits a natural topological interpretation; and we design a calculus consisting of finitely many axioms and rules that is sound and complete with respect to a category of algebras which is dual to the category KHaus of compact Hausdorff spaces and continuous maps. Our framework is built on de Vries duality, which we will now discuss in more detail.

A subordination on a Boolean algebra B is a binary relation $\prec$ on B satisfying certain conditions (see Definition 2.1). Subordinations were introduced in [6]. They are in one-to-one correspondence with quasi-modal operators of [9] and pre-contact relations of [13]. Subordinations can be modeled dually by closed relations on Stone spaces. This leads to a duality between the category Sub of subordination algebras and the category StR of pairs (X, R) where X is a Stone space and R is a closed relation on X (see [6, Sec. 2.1]). Further conditions on $\prec$ characterize when the relation R is reflexive or symmetric. This yields the subcategories RSub and Con of Sub consisting of reflexive subordination algebras and contact algebras, respectively. As the name suggests, reflexive subordinations dually correspond to reflexive closed relations; while contact relations correspond to reflexive and symmetric closed relations.

Compingent algebras, introduced by de Vries [12], are obtained by adding two additional conditions to the definition of contact algebras. As was shown in [6, Lem. 6.3], they are dually characterized by irreducible equivalence relations (the definition is given in Section 2). Thus, the category Com of compingent algebras is dually equivalent to the subcategory of StR consisting of the pairs (X, R) where R is an irreducible equivalence relation. A de Vries algebra is a complete compingent algebra. Since complete Boolean algebras dually correspond to extremally disconnected Stone spaces, we conclude that the category DeV of de Vries algebras is dually equivalent to the subcategory of StR consisting of the pairs (X, R) where the Stone space X is extremally disconnected and R is an irreducible equivalence relation. Such pairs were called Gleason spaces in [6] since they are closely related to Gleason covers of compact Hausdorff spaces. A key result of [6, Thm. 6.13] is that this close correspondence between Gleason spaces and Gleason covers yields that the category Gle of Gleason spaces is equivalent to KHaus. Since Gle is dually equivalent to DeV, we arrive at de Vries duality: KHaus is dually equivalent to DeV.

It was pointed out in [6, Sec. 3] that subordinations on a Boolean algebra B can alternatively be described by binary operations on B called strict implications. In this paper we study the resulting variety of strict implication algebras. The study simplifies considerably if we work with the strict implications that correspond to reflexive subordinations. In Section 3 we prove that the resulting variety SIA of strict implication algebras is a discriminator variety, and we give its axiomatization. We also prove that SIA is a locally finite variety. In Section 4 we develop the corresponding strict implication calculus SIC, which is a modal logic with one binary modality that corresponds to the strict implication. Section 5 is devoted to the symmetric strict implication calculus $\mathsf{S^2IC}$ which is an extension of SIC. The corresponding variety $\mathsf{S^2IA}$ is the subvariety of SIA generated by the strict implication algebras that correspond to contact algebras. One of our main results is that $\mathsf{S^2IA}$ is generated by the strict implication algebras that correspond to de Vries algebras. This yields that $\mathsf{S^2IC}$ is complete with respect to DeV, which coupled with de Vries duality, yields that $\mathsf{S^2IC}$ is the logic of compact Hausdorff spaces.

Our approach is closely related to that of Balbiani et al. [1], which along with [6] inspired the current paper. Balbiani et al. develop two-sorted logical calculi for region-based theories of space. Our calculus is simpler as we work with one-sorted propositional modal logic with one binary modality. In Section 7 we show how to translate the language of [1] into our language.

Two of the defining axioms of de Vries algebras are universal-existential statements or Π_2-statements. They can be expressed in our language by use of non-standard rules, which we call Π_2-rules. That $\mathsf{S^2IC}$ is complete with respect to de Vries algebras shows that these Π_2-rules are admissible in $\mathsf{S^2IC}$. This is again closely related to Balbiani et al. [1] who use similar non-standard rules in the context of region-based theories of space and prove their admissibility. In Section 6 we develop the theory of Π_2-rules, show that they define inductive elementary subclasses of RSub, and that every derivation system axiomatized by Π_2-rules is strongly sound and complete with respect to the subclass of RSub it defines. We also give a criterion of when a Π_2-rule is admissible. We prove that being a zero-dimensional de Vries algebra is definable by a Π_2-rule, and that this rule is admissible in $\mathsf{S^2IC}$. As a consequence, we obtain that $\mathsf{S^2IC}$ is complete with respect to zero-dimensional de Vries algebra, and hence with respect to zero-dimensional compact Hausdorff spaces, also known as Stone spaces. On the other side of the spectrum from zero-dimensional spaces are connected spaces. We define the connected symmetric strict implication calculus $\mathsf{CS^2IC}$ by adding one axiom to $\mathsf{S^2IC}$, and prove that $\mathsf{CS^2IC}$ is complete with respect to connected de Vries algebras, and consequently with respect to connected compact Hausdorff spaces.

2. Subordinations, contact algebras, and de Vries algebras

In this section we recall the definitions of subordination, contact algebra, compingent algebra, and de Vries algebra, as well as the duality theory for these algebras. We also connect the duality theory for de Vries algebras to de Vries duality for compact Hausdorff spaces via Gleason spaces.

Definition 2.1. ([6])

(1) A *subordination* on a Boolean algebra B is a binary relation $\prec$ satisfying:
   - (S1) $0 \prec 0$ and $1 \prec 1$;
   - (S2) $a \prec b, c$ implies $a \prec b \wedge c$;
   - (S3) $a, b \prec c$ implies $a \vee b \prec c$;
   - (S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

(2) We call $(B, \prec)$ a *subordination algebra*, and let Sub be the class of all subordination algebras.

By Stone duality, Boolean algebras correspond to zero-dimensional compact Hausdorff spaces, known as *Stone spaces*. Given a Boolean algebra B, its dual Stone space is the space X of ultrafilters of B, the topology on which is given by the basis $\{\beta(a) \mid a \in B\}$, where $\beta(a) = \{x \in X \mid a \in x\}$. Then β is an isomorphism from B to the Boolean algebra $\mathsf{Clop}(X)$ of clopen subsets of X.

We say that a binary relation R on a Stone space X is *closed* if R is a closed subset of $X \times X$ in the product topology. Let StR be the class of pairs (X, R) where X is a Stone space and R is a closed relation on X. There is a one-to-one correspondence between Sub and StR, which extends to a categorical duality; see [6, Sec. 2.1]. This one-to-one correspondence can be obtained as follows. As usual, for a binary relation R on a set X and $S \subseteq X$, we write

$$R[S] := \{x \in X \mid sRx \text{ for some } s \in S\}.$$

Let B be a Boolean algebra and X the Stone space of B. If R is a closed relation on X, then the binary relation $\prec$ defined by $a \prec b$ iff $R[\beta(a)] \subseteq \beta(b)$ is a subordination on B. Conversely, let $\prec$ be a subordination on B. For $S \subseteq B$, write

$$\uparrow S := \{a \in B \mid s \prec a \text{ for some } s \in S\},$$

and define a binary relation R on X by xRy iff $\uparrow x \subseteq y$. Then R is a closed relation on X, and this correspondence is one-to-one.

We next consider the following additional properties of $\prec$:

(S5) $a \prec b$ implies $a \leq b$;
(S6) $a \prec b$ implies $\neg b \prec \neg a$;
(S7) $a \prec b$ implies there is $c \in B$ with $a \prec c \prec b$;
(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

The next lemma gives a dual characterization of (S5)-(S7).

Lemma 2.2 ([13]). *Let B be a Boolean algebra, X the Stone space of B, $\prec$ a subordination on B, and R the corresponding closed relation on X.*

(1) $(B, \prec)$ *satisfies* (S5) *iff R is reflexive.*
(2) $(B, \prec)$ *satisfies* (S6) *iff R is symmetric.*
(3) $(B, \prec)$ *satisfies* (S7) *iff R is transitive.*

Definition 2.3. Let $(B, \prec)$ be a subordination algebra.

(1) We call $(B, \prec)$ *reflexive* if $(B, \prec)$ satisfies (S5), and let RSub be the class of reflexive subordination algebras.[2]
(2) ([27]) We call $(B, \prec)$ a *contact algebra* if $(B, \prec)$ satisfies (S5) and (S6), and let Con be the class of contact algebras.

We clearly have that $\mathsf{Con} \subset \mathsf{RSub} \subset \mathsf{Sub}$, that reflexive subordination algebras dually correspond to the subclass of StR consisting of reflexive closed relations on Stone spaces, and that contact algebras dually correspond to the subclass of StR consisting of reflexive and symmetric closed relations on Stone spaces.

Definition 2.4.

(1) ([12]) We call a contact algebra $(B, \prec)$ a *compingent algebra* if it satisfies (S7) and (S8).
(2) ([4]) We call a compingent algebra $(B, \prec)$ a *de Vries algebra* if B is a complete Boolean algebra.
(3) Let Com be the class of compingent algebras and DeV the class of de Vries algebras.

We clearly have that $\mathsf{DeV} \subset \mathsf{Com} \subset \mathsf{Con}$. Let $(B, \prec)$ be a contact algebra and let (X, R) be its dual. As follows from Lemma 2.2, R is a reflexive and symmetric closed relation. Moreover, $(B, \prec)$ satisfies (S7) iff R is an equivalence relation. By [6, Lem. 6.3], $(B, \prec)$ satisfies (S8) iff R is an irreducible equivalence relation, where we recall (see [6, Def. 6.1 and Rem. 6.2]) that R is *irreducible* provided $R[U]$ is a proper subset of X for each proper clopen subset U of X.

To characterize dually de Vries algebras we recall that a Boolean algebra B is complete iff its Stone space X is extremally disconnected, where a space is *extremally disconnected* provided the closure of each open set is clopen. Thus, compingent algebras dually correspond to pairs (X, R) where X is a Stone space and R is an irreducible equivalence relation, while de Vries algebras correspond to pairs (X, R) where X is an extremally disconnected Stone space and R is an irreducible equivalence relation. Such pairs were

[2] We point out that $(B, \prec)$ being reflexive does not mean that $\prec$ is a reflexive relation, rather that the corresponding closed relation on the Stone space is reflexive (see Lemma 2.2(1)).

called *Gleason spaces* in [6, Def. 6.6] because of the close connection to Gleason covers of compact Hausdorff spaces.[3]

Let X be a compact Hausdorff space, and let (Y, π) be the Gleason cover of X. Define R on Y by xRy iff $\pi(x) = \pi(y)$. Then (Y, R) is a Gleason space. Conversely, if (Y, R) is a Gleason space, then the quotient space $X := Y/R$ is compact Hausdorff. This establishes a one-to-one correspondence between Gleason spaces and compact Hausdorff spaces, which extends to a categorical duality (see [6, Sec. 6] for details).

Since DeV dually corresponds to the class of Gleason spaces, it follows that DeV dually corresponds to the class KHaus of compact Hausdorff spaces, which is the object level of the celebrated de Vries duality [12]. The correspondence between DeV and KHaus can be obtained directly, as was done by de Vries.

For a compact Hausdorff space X, let $\mathcal{RO}(X)$ be the complete Boolean algebra of regular open subsets of X. Define $\prec$ on $\mathcal{RO}(X)$ by

$$U \prec V \text{ iff } \mathsf{Cl}(U) \subseteq V.$$

Then $(\mathcal{RO}(X), \prec)$ is a de Vries algebra (that it validates (S7) and (S8) follows from the fact that every compact Hausdorff space is regular and normal; see, e.g., [14, Sec. 3.1]).

Conversely, suppose $(B, \prec)$ is a compingent algebra. A *round filter* of $(B, \prec)$ is a filter F of B satisfying $\twoheaduparrow F = F$. An *end* of $(B, \prec)$ is a maximal proper round filter. Let X be the set of ends of $(B, \prec)$. For $a \in B$, let $\beta(a) = \{x \in X \mid a \in x\}$. Then $\{\beta(a) \mid a \in B\}$ generates a compact Hausdorff topology on X. Moreover, if X is compact Hausdorff, then it is homeomorphic to the dual of $(\mathcal{RO}(X), \prec)$. If $(B, \prec)$ is a compingent algebra and X is its dual, then $(B, \prec)$ embeds into $(\mathcal{RO}(X), \prec)$, and $(B, \prec)$ is isomorphic to $(\mathcal{RO}(X), \prec)$ iff $(B, \prec)$ is a de Vries algebra. These correspondences extend to contravariant functors, which yield a dual equivalence of the categories KHaus and DeV. We refer to [12] for missing details and proofs.

3. The variety of strict implication algebras

As was pointed out in [6, Sec. 3], subordinations on B can be described by means of binary operations $\rightsquigarrow: B \times B \to B$ with values in $\{0, 1\}$ satisfying

(I1) $0 \rightsquigarrow a = a \rightsquigarrow 1 = 1$;
(I2) $(a \vee b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$;
(I3) $a \rightsquigarrow (b \wedge c) = (a \rightsquigarrow b) \wedge (a \rightsquigarrow c)$.

If $\prec$ is a subordination on B, then define $\rightsquigarrow: B \times B \to B$ by

$$a \rightsquigarrow b = \begin{cases} 1 & \text{if } a \prec b \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\rightsquigarrow$ has values in $\{0, 1\}$ and satisfies (I1)–(I3). Conversely, given $\rightsquigarrow$, define $\prec$ by setting

$$a \prec b \text{ iff } a \rightsquigarrow b = 1.$$

It is easy to see that $\prec$ is a subordination on B, and that this correspondence is one-to-one.

Moreover, the axioms (S5)–(S8) correspond, respectively, to the axioms:

(I4) $a \rightsquigarrow b \leq a \to b$;
(I5) $a \rightsquigarrow b = \neg b \rightsquigarrow \neg a$;
(I6) $a \rightsquigarrow b = 1$ implies $\exists c : a \rightsquigarrow c = 1$ and $c \rightsquigarrow b = 1$;

[3] For details on Gleason covers we refer to [19] and [22, Sec. III.3]. They are not crucial for the content of this paper.

(I7) $a \neq 0$ implies $\exists b \neq 0 : b \rightsquigarrow a = 1$.

Note that (I2)-(I3) correspond to (S2)-(S4) which explains why the numbering of the I-axioms is one off the numbering of the S-axioms. As we will see, adding (I4) to (I1)–(I3) is very useful in algebraic as well as logical calculations. Therefore, as our base variety, we will consider the variety generated by the algebras $(B, \rightsquigarrow)$, where B is a Boolean algebra and $\rightsquigarrow$ is a binary operation on B with values in $\{0, 1\}$ satisfying (I1)–(I4). From now on, when we write $(B, \rightsquigarrow) \in \mathsf{RSub}$, we mean that the corresponding $(B, \prec)$ is reflexive (see Definition 2.3(1)).

Definition 3.1. We call $(B, \rightsquigarrow)$ a *strict implication algebra* if $(B, \rightsquigarrow)$ belongs to the variety generated by RSub. Let SIA be the variety of strict implication algebras.

Remark 3.2. While $\rightsquigarrow$ is not a normal and additive operator on B, it gives rise to the normal and additive operator $\Delta(a, b) := \neg(a \rightsquigarrow \neg b)$. Then (B, Δ) is a BAO (Boolean algebra with operators), and $\rightsquigarrow$ is definable from Δ by $a \rightsquigarrow b = \neg\Delta(a, \neg b)$. We prefer to work with $\rightsquigarrow$ since it arises from subordinations more naturally.

Let $(B, \rightsquigarrow) \in \mathsf{SIA}$. For $a \in B$, define

$$\Box a = 1 \rightsquigarrow a.$$

By (I1), $\Box 1 = 1 \rightsquigarrow 1 = 1$; and by (I3), $\Box(a \wedge b) = 1 \rightsquigarrow (a \wedge b) = (1 \rightsquigarrow a) \wedge (1 \rightsquigarrow b) = \Box a \wedge \Box b$. Thus, $(B, \Box)$ is a (normal) modal algebra.

Suppose $(B, \rightsquigarrow) \in \mathsf{RSub}$. If $a = 1$, then $\Box a = 1$. If $a \neq 1$, then by (I4), $\Box a = 1 \rightsquigarrow a \leq 1 \to a = a \neq 1$, so $\Box a \neq 1$. But since $(B, \rightsquigarrow) \in \mathsf{RSub}$, we have that $\rightsquigarrow$ only takes values in $\{0, 1\}$. Therefore, $\Box$ only takes values in $\{0, 1\}$. Thus, $\Box a = 0$, and so

$$\Box a = \begin{cases} 1 \text{ if } a = 1 \\ 0 \text{ if } a \neq 1. \end{cases}$$

We let $\Diamond$ be the dual of $\Box$, i.e., $\Diamond a = \neg\Box\neg a$. Then

$$\Diamond a = \begin{cases} 0 \text{ if } a = 0 \\ 1 \text{ if } a \neq 0, \end{cases}$$

and so $\Diamond$ is the so-called *unary discriminator term* [21]. From this, and the fact that the class RSub is axiomatized by universal first-order formulas, the following observations are immediate [29, Sec. 8.2.].

Proposition 3.3.

(1) *The variety* SIA *is a discriminator variety, and hence a semisimple variety.*
(2) *The simple algebras in* SIA *are exactly the members of* RSub.

We next turn to the axiomatization of SIA. First we observe that $\Box a \leq a$ by (I4). Let $\mathcal{V}$ be the variety of algebras $(B, \rightsquigarrow)$ axiomatized by the equations defining Boolean algebras, (I1)–(I4), and the axioms:

(I8) $\Box a \leq \Box\Box a$;
(I9) $\neg\Box a \leq \Box\neg\Box a$;
(I10) $a \rightsquigarrow b = \Box(a \rightsquigarrow b)$;
(I11) $\Box a \leq \neg\Box a \rightsquigarrow 0$.

We recall that a modal algebra $(B, \Box)$ is an S5-*algebra* if its satisfies $\Box a \le a$, $\Box a \le \Box\Box a$, and $\neg\Box a \le \Box\neg\Box a$. Thus, if $(B, \rightsquigarrow) \in \mathcal{V}$, then $(B, \Box)$ is an S5-algebra.

Theorem 3.4. *SIA* $= \mathcal{V}$.

Proof. It is straightforward to see that (I8)–(I11) hold in each member of RSub. Since SIA is generated by RSub, it follows that SIA $\subseteq \mathcal{V}$. For the reverse inclusion, we utilize [21, Thm. 3], by which a unary term $\Diamond$ is a discriminator term in subdirectly irreducible members of a variety $\mathcal{V}$ iff $\mathcal{V}$ satisfies four equations that in our setting amount to:

- $\Box a \le \Box\Box a$;
- $\neg\Box a \le \Box\neg\Box a$;
- $\Box a \le \neg\Box a \rightsquigarrow 0$;
- $\neg\Box a \le \Box a \rightsquigarrow 0$.

Clearly each $(B, \rightsquigarrow) \in \mathcal{V}$ satisfies the first three. To see that it also satisfies the fourth, observe that by (I4) and (I9) we have $\neg\Box a = \Box\neg\Box a$. This together with (I11) yields

$$\neg\Box a = \Box\neg\Box a \le \neg\Box\neg\Box a \rightsquigarrow 0 = \neg\neg\Box a \rightsquigarrow 0 = \Box a \rightsquigarrow 0.$$

Thus, [21, Thm. 3] applies, by which $\Diamond$ is a unary discriminator in all subdirectly irreducible members of $\mathcal{V}$. So if $(B, \rightsquigarrow)$ is a subdirectly irreducible member of $\mathcal{V}$, then $\Box$ only takes the values 0 and 1. Since (I10) holds in $(B, \rightsquigarrow)$, we have that $\rightsquigarrow$ also only takes the values 0 and 1. Therefore, as (I1)–(I4) hold in $(B, \rightsquigarrow)$, it follows from the definition of RSub that $(B, \rightsquigarrow) \in$ RSub. Thus, each subdirectly irreducible member of $\mathcal{V}$ belongs to SIA. Since $\mathcal{V}$ is generated by its subdirectly irreducible algebras, we conclude that $\mathcal{V} \subseteq$ SIA. $\Box$

We next give an alternate axiomatization of SIA, which will be useful in Section 4. Let $\mathcal{W}$ be the variety axiomatized by (I1)–(I4) and the axioms:

(I12) $\Box(a \to b) \wedge (b \rightsquigarrow c) \le a \rightsquigarrow c$;
(I13) $(a \rightsquigarrow b) \wedge \Box(b \to c) \le a \rightsquigarrow c$;
(I14) $a \rightsquigarrow b \le c \rightsquigarrow (a \rightsquigarrow b)$;
(I15) $\neg(a \rightsquigarrow b) \le c \rightsquigarrow \neg(a \rightsquigarrow b)$.

To prove that SIA $= \mathcal{W}$, we require the following lemma.

Lemma 3.5. (I2) *and* (I3) *imply* $a \le b \Rightarrow (b \rightsquigarrow c \le a \rightsquigarrow c$ *and* $c \rightsquigarrow a \le c \rightsquigarrow b)$.

Proof. By (I2), $b \rightsquigarrow c = (a \vee b) \rightsquigarrow c = (a \rightsquigarrow c) \wedge (b \rightsquigarrow c)$. Therefore, $b \rightsquigarrow c \le a \rightsquigarrow c$. Also, by (I3), $c \rightsquigarrow a = c \rightsquigarrow (a \wedge b) = (c \rightsquigarrow a) \wedge (c \rightsquigarrow b)$. Thus, $c \rightsquigarrow a \le c \rightsquigarrow b$. $\Box$

Theorem 3.6. *SIA* $= \mathcal{W}$.

Proof. First we show that SIA $\subseteq \mathcal{W}$. For this it is sufficient to see that (I12)–(I15) hold in each strict implication algebra $(B, \rightsquigarrow)$. To see that (I12) holds, by (I11),

$$\Box(a \to b) \wedge (b \rightsquigarrow c) \le (\neg\Box(a \to b) \rightsquigarrow 0) \wedge (b \rightsquigarrow c).$$

Since $0 \le c$, by Lemma 3.5 and (I2),

$$\begin{aligned}(\neg\Box(a\to b)\rightsquigarrow 0)\wedge(b\rightsquigarrow c)&\le(\neg\Box(a\to b)\rightsquigarrow c)\wedge(b\rightsquigarrow c)\\&=(\neg\Box(a\to b)\vee b)\rightsquigarrow c\\&=(\Box(a\to b)\to b)\rightsquigarrow c.\end{aligned}$$

Because $\Box(a\to b)\le a\to b$, we have $(a\to b)\to b\le\Box(a\to b)\to b$. Therefore, by Lemma 3.5,

$$(\Box(a\to b)\to b)\rightsquigarrow c\le((a\to b)\to b)\rightsquigarrow c.$$

But $(a\to b)\to b=\neg(\neg a\vee b)\vee b=a\vee b$, so applying Lemma 3.5 again yields

$$((a\to b)\to b)\rightsquigarrow c=(a\vee b)\rightsquigarrow c\le a\rightsquigarrow c.$$

Thus, (I12) holds in $(B,\rightsquigarrow)$.

To see that (I13) holds, by Lemma 3.5 and (I3),

$$\begin{aligned}(a\rightsquigarrow b)\wedge\Box(b\to c)&=(a\rightsquigarrow b)\wedge(1\rightsquigarrow(b\to c))\\&\le(a\rightsquigarrow b)\wedge(a\rightsquigarrow(b\to c))\\&=a\rightsquigarrow(b\wedge(b\to c)).\end{aligned}$$

Since $b\wedge(b\to c)\le c$, applying Lemma 3.5 again yields

$$a\rightsquigarrow(b\wedge(b\to c))\le a\rightsquigarrow c.$$

Thus, (I13) holds in $(B,\rightsquigarrow)$.

To see that (I14) holds, by (I10) and Lemma 3.5,

$$a\rightsquigarrow b=\Box(a\rightsquigarrow b)=1\rightsquigarrow(a\rightsquigarrow b)\le c\rightsquigarrow(a\rightsquigarrow b).$$

Thus, (I14) holds in $(B,\rightsquigarrow)$.

To see that (I15) holds, since $(B,\Box)$ is an $\mathsf{S5}$-algebra, by (I10) and Lemma 3.5,

$$\begin{aligned}\neg(a\rightsquigarrow b)&=\neg\Box(a\rightsquigarrow b)=\Box\neg\Box(a\rightsquigarrow b)\\&=\Box\neg(a\rightsquigarrow b)=1\rightsquigarrow\neg(a\rightsquigarrow b)\\&\le c\rightsquigarrow\neg(a\rightsquigarrow b).\end{aligned}$$

Thus, (I15) holds in $(B,\rightsquigarrow)$. Consequently, $\mathsf{SIA}\subseteq\mathcal{W}$.

It is left to show that $\mathcal{W}\subseteq\mathsf{SIA}$. For this it is sufficient to see that (I8)–(I11) hold in each $(B,\rightsquigarrow)\in\mathcal{W}$. It follows from (I14) that $a\rightsquigarrow b\le\Box(a\rightsquigarrow b)$, and it follows from (I4) that $\Box(a\rightsquigarrow b)\le a\rightsquigarrow b$. Thus, (I10) holds in $(B,\rightsquigarrow)$.

That (I8) holds in $(B,\rightsquigarrow)$ is immediate from (I10):

$$\Box a=1\rightsquigarrow a=\Box(1\rightsquigarrow a)=\Box\Box a.$$

To see that (I11) holds, substituting in (I12) $\neg\Box a$ for a and 0 for both b and c yields $\Box(\neg\Box a\to 0)\wedge(0\rightsquigarrow 0)\le\neg\Box a\rightsquigarrow 0$. Now, using (I1) and (I8), we have:

$$\Box(\neg\Box a\to 0)\wedge(0\rightsquigarrow 0)=\Box(\neg\neg\Box a)\wedge 1=\Box\Box a=\Box a.$$

Thus, $\Box a \le \neg\Box a \rightsquigarrow 0$, and so (I11) holds in $(B, \rightsquigarrow)$.

Finally, it follows from (I15) that $\neg(a \rightsquigarrow b) \le \Box\neg(a \rightsquigarrow b)$. Thus,

$$\neg\Box a = \neg(1 \rightsquigarrow a) \le \Box\neg(1 \rightsquigarrow a) = \Box\neg\Box a,$$

and hence (I9) holds in $(B, \rightsquigarrow)$. Consequently, $\mathcal{W} \subseteq \mathsf{SIA}$. □

We next show that our base variety SIA is locally finite, and consider subvarieties and inductive subclasses of SIA.

Proposition 3.7. *The variety* SIA *is locally finite.*

Proof. Let $(B, \rightsquigarrow) \in \mathsf{RSub}$ be n-generated, with generators $a_1, \dots, a_n \in B$. For each $a \in B$, there is a term $t(x_1, \dots, x_n)$ such that $a = t(a_1, \dots, a_n)$. Since $(B, \rightsquigarrow) \in \mathsf{RSub}$, for each $b, c \in B$, we have $b \rightsquigarrow c \in \{0, 1\}$. Therefore, by replacing each subterm of $t(x_1, \dots, x_n)$ of the form $x \rightsquigarrow y$ with either 0 or 1, we obtain a Boolean term $t'(x_1, \dots, x_n)$ such that $a = t'(a_1, \dots, a_n)$. Thus, B is n-generated as a Boolean algebra, and hence has at most 2^{2^n} elements. Since RSub is the class of simple algebras in SIA, which is a semisimple variety (see Proposition 3.3), there is a uniform bound $m(n) = 2^{2^n}$ on all n-generated subdirectly irreducible members of SIA. Consequently, by [3, Thm. 3.7(4)], SIA is locally finite. □

As an immediate consequence we obtain:

Corollary 3.8. *Every subvariety of* SIA *is generated by its finite members.*

While SIA has many subvarieties, we will be interested in the subvariety obtained by postulating the identity (I5). Our interest is motivated by the fact that this variety is exactly the subvariety of SIA generated by the class Con of contact algebras. We further restrict Con to the class Com of compingent algebras, by postulating (I6) and (I7). But unlike (I5), neither (I6) nor (I7) is an identity. However, both (I6) and (I7) are Π_2-statements (i.e., statements of the form $\forall\overline{x}\exists\overline{y}\Phi(\overline{x}, \overline{y})$, where $\overline{x}, \overline{y}$ are tuples of variables and $\Phi(\overline{x}, \overline{y})$ is a quantifier-free formula). By the Chang-Łoś-Suszko Theorem (see, e.g., [11, Thm. 3.2.3]), the elementary classes corresponding to Π_2-statements are inductive classes, where we recall that a class is *inductive* provided it is closed under unions of chains (equivalently, closed under directed limits). While we will be mainly interested in the inductive class Com, in Section 6 we will show that all elementary inductive subclasses of RSub can be axiomatized by non-standard rules.

We conclude this section by observing that, unlike subvarieties of SIA, not every inductive subclass of SIA is determined by its finite algebras. For example, the inductive elementary class Com is not determined by its finite algebras. To see this, let Dis be the subclass of Com consisting of those algebras in Com that validate the equation $a \rightsquigarrow a = 1$. Then Dis is an inductive elementary subclass of Com. To see that Dis is a proper subclass of Com, let $X = [0, 1]$. Then the de Vries algebra $(\mathcal{RO}(X), \prec)$ falsifies $a \rightsquigarrow a = 1$. Indeed, if we put $a = [0, \frac{1}{2})$, then the closure of a is $[0, \frac{1}{2}] \nsubseteq a$. So $a \not\prec a$, and hence $a \rightsquigarrow a \ne 1$. On the other hand, we show that every finite algebra in Com validates the equation $a \rightsquigarrow a = 1$. Since every finite compingent algebra is a finite de Vries algebra, by de Vries duality, every finite compingent algebra $(B, \prec)$ is isomorphic to the powerset of a finite discrete space X. Because every subset of a discrete space is clopen, we have $a \prec a$, so $a \rightsquigarrow a = 1$ for each $a \in B$. Therefore, Dis and Com have the same finite algebras, yet Dis is a proper subclass of Com. Thus, Com is not determined by its finite algebras.

4. The strict implication calculus

We next present a sound and complete deductive system for SIA. We will work with the language of classical propositional logic, with a countably infinite supply of propositional letters and primitive connectives

$\wedge, \neg$, which we will enrich with one binary connective $\rightsquigarrow$ of *strict implication*. Then $\top, \bot, \vee, \to, \leftrightarrow$ are usual abbreviations, and $\Box\varphi$ abbreviates $\top \rightsquigarrow \varphi$.

A *valuation* on $(B, \rightsquigarrow)$ is an assignment of elements of B to propositional letters of our language $\mathcal{L}$, which extends to all formulas of $\mathcal{L}$ in the usual way. We say that a valuation v on $(B, \rightsquigarrow)$ *satisfies* a formula φ if $v(\varphi) = 1$. In such a case we write $(B, \rightsquigarrow, v) \models \varphi$. If all valuations on $(B, \rightsquigarrow)$ satisfy φ, then we say that $(B, \rightsquigarrow)$ *validates* φ, and write $(B, \rightsquigarrow) \models \varphi$. For a set of formulas Γ, we write $(B, \rightsquigarrow) \models \Gamma$ if $(B, \rightsquigarrow) \models \varphi$ for every $\varphi \in \Gamma$.

Suppose $\mathcal{U} \subseteq \mathsf{SIA}$, φ is a formula, and Γ is a set of formulas. We say that φ is a *semantic consequence* of Γ over $\mathcal{U}$, and write $\Gamma \models_{\mathcal{U}} \varphi$, provided for each $(B, \rightsquigarrow) \in \mathcal{U}$ and each valuation v on $(B, \rightsquigarrow)$, if $v(\gamma) = 1$ for each $\gamma \in \Gamma$, then $v(\varphi) = 1$.

Consider the following axiom schemes:

(A1) $(\bot \rightsquigarrow \varphi) \wedge (\varphi \rightsquigarrow \top)$,
(A2) $[(\varphi \vee \psi) \rightsquigarrow \chi] \leftrightarrow [(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi)]$,
(A3) $[\varphi \rightsquigarrow (\psi \wedge \chi)] \leftrightarrow [(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi)]$,
(A4) $(\varphi \rightsquigarrow \psi) \to (\varphi \to \psi)$,
(A5) $(\varphi \rightsquigarrow \psi) \leftrightarrow (\neg\psi \rightsquigarrow \neg\varphi)$,
(A8) $\Box\varphi \to \Box\Box\varphi$,
(A9) $\neg\Box\varphi \to \Box\neg\Box\varphi$,
(A10) $(\varphi \rightsquigarrow \psi) \leftrightarrow \Box(\varphi \rightsquigarrow \psi)$,
(A11) $\Box\varphi \to (\neg\Box\varphi \rightsquigarrow \bot)$,
(A12) $[\Box(\varphi \to \psi) \wedge (\psi \rightsquigarrow \chi)] \to (\varphi \rightsquigarrow \chi)$,
(A13) $[(\varphi \rightsquigarrow \psi) \wedge \Box(\psi \to \chi)] \to (\varphi \rightsquigarrow \chi)$,
(A14) $(\varphi \rightsquigarrow \psi) \to [\chi \rightsquigarrow (\varphi \rightsquigarrow \psi)]$,
(A15) $\neg(\varphi \rightsquigarrow \psi) \to [\chi \rightsquigarrow \neg(\varphi \rightsquigarrow \psi)]$.

Clearly (A1)–(A5) correspond to (I1)–(I5) and (A8)–(A15) to (I8)–(I15).

Definition 4.1. The *strict implication calculus* SIC is the derivation system containing:

- all the theorems of the classical propositional calculus CPC,
- the axiom schemes (A1)–(A5) and (A8)–(A11),

and closed under the inference rules:

(MP) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

(N) $\dfrac{\varphi}{\Box\varphi}$

The definition of derivability in SIC is standard:

Definition 4.2.

(1) A *proof* of a formula φ from a set of formulas Γ is a finite sequence $\psi_1, \dots, \psi_n$ such that $\psi_n = \varphi$ and each ψ_i is in Γ or is an instance of an axiom of SIC or is obtained from ψ_j, ψ_k for some $j, k < i$ by applying (MP), or is obtained from ψ_j for some $j < i$ by applying (N). Elements of Γ are referred to as *assumptions*.
(2) If there is a proof of φ from Γ, then we say that φ is *derivable in* SIC *from* Γ and write $\Gamma \vdash_{\mathsf{SIC}} \varphi$.
(3) If $\Gamma = \varnothing$, then we say that φ is *derivable in* SIC and write $\vdash_{\mathsf{SIC}} \varphi$.

Remark 4.3. Since $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$ is an instance of (A3) and $\Box\varphi \to \varphi$ is an instance of (A4), we see that all the theorems of the modal system S5 are derivable in SIC. In particular, the (K) axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is derivable in SIC.

The deduction theorem for SIC is proved as for S5:

Theorem 4.4. *For any set of formulas Γ and for any formulas φ, ψ, we have:*

$$\Gamma \cup \{\varphi\} \vdash_{\mathsf{SIC}} \psi \quad \Leftrightarrow \quad \Gamma \vdash_{\mathsf{SIC}} \Box\varphi \to \psi.$$

Proof. ($\Leftarrow$) This is the easy direction since a proof $\psi_1, \dots, \psi_n$ of $\Box\varphi \to \psi$ from Γ can easily be extended to a proof of ψ from $\Gamma \cup \{\varphi\}$ as follows:

n. $\Box\varphi \to \psi$
$n+1$. φ (assumption)
$n+2$. $\Box\varphi$ (by (N) from $n+1$)
$n+3$. ψ (by (MP) from n and $n+2$).

($\Rightarrow$) Suppose there is a proof $\psi_1, \dots, \psi_n$ of ψ from $\Gamma \cup \{\varphi\}$. We show by induction on $i = 1, \dots, n$ that we can obtain a proof of $\Box\varphi \to \psi_i$ from Γ. If $\psi_i = \varphi$, then $\Gamma \vdash_{\mathsf{SIC}} \Box\varphi \to \psi_i$ since $\vdash_{\mathsf{SIC}} \Box\varphi \to \varphi$. If $\psi_i \in \Gamma$ or ψ_i is an instance of an axiom of SIC, then since $\vdash_{\mathsf{SIC}} \psi_i \to (\Box\varphi \to \psi_i)$, by applying (MP) we obtain $\Gamma \vdash_{\mathsf{SIC}} \Box\varphi \to \psi_i$. If ψ_i is obtained by applying (MP) to ψ_j and $\psi_k = \psi_j \to \psi_i$ with $j, k < i$, then by the inductive hypothesis, $\Gamma \vdash_{\mathsf{SIC}} \Box\varphi \to \psi_j, \Box\varphi \to (\psi_j \to \psi_i)$. But then $\Gamma \vdash_{\mathsf{SIC}} \Box\varphi \to \psi_j, \psi_j \to (\Box\varphi \to \psi_i)$, which yields $\Gamma \vdash_{\mathsf{SIC}} \Box\varphi \to (\Box\varphi \to \psi_i)$, so $\Gamma \vdash_{\mathsf{SIC}} \Box\varphi \to \psi_i$. Finally, if ψ_i is obtained by applying (N) to ψ_j with $j < i$, then by the inductive hypothesis, $\Gamma \vdash_{\mathsf{SIC}} \Box\varphi \to \psi_j$. Applying (N) yields $\Gamma \vdash_{\mathsf{SIC}} \Box(\Box\varphi \to \psi_i)$. Therefore, by applying the (K) axiom (see Remark 4.3) and (MP), we obtain $\Gamma \vdash_{\mathsf{SIC}} \Box\Box\varphi \to \Box\psi_j$. Thus, since $\vdash_{\mathsf{SIC}} \Box\varphi \to \Box\Box\varphi$, we have $\Gamma \vdash_{\mathsf{SIC}} \Box\varphi \to \Box\psi_j$, and so $\Gamma \vdash_{\mathsf{SIC}} \Box\varphi \to \psi_i$, concluding the proof. □

Since each axiom of SIC has an equational counterpart in the axiomatization of SIA, the standard Lindenbaum construction (see, e.g., [23]) yields the following.

Proposition 4.5. SIC *is strongly sound and complete with respect to* SIA*; that is, for a set of formulas Γ and a formula φ,*

$$\Gamma \vdash_{\mathsf{SIC}} \varphi \textit{ iff } \Gamma \models_{\mathsf{SIA}} \varphi.$$

Remark 4.6. As follows from Theorem 3.6, SIA can be axiomatized by replacing (I8)–(I11) with (I12)–(I15). Thus, by Proposition 4.5, SIC can be axiomatized by replacing (A8)–(A11) with (A12)–(A15).

We next show that SIC is in fact strongly sound and complete with respect to RSub. For this we first characterize congruences of strict implication algebras. It is well known that congruences of Boolean algebras correspond to filters, and this correspondence is obtained as follows. If θ is a congruence on a Boolean algebra B, then $F_\theta = \{a \in B \mid a\theta 1\}$ is a filter of B. If F is a filter of B, then θ_F defined by $a\theta_F b$ iff $a \leftrightarrow b \in F$ is a congruence of B. Moreover, $\theta_{F_\theta} = \theta$ and $F_{\theta_F} = F$.

Proposition 4.7. *For $(B, \rightsquigarrow) \in$ SIA, there is a one-to-one correspondence between*

(1) *congruences of $(B, \rightsquigarrow)$;*
(2) *congruences θ of B such that $a\theta b$ implies $(a \rightsquigarrow c)\theta(b \rightsquigarrow c)$ and $(c \rightsquigarrow a)\theta(c \rightsquigarrow b)$;*
(3) *filters F of B such that $a \in F$ implies $\Box a \in F$;*

(4) *filters* F *of* B *such that* $a \to b \in F$ *implies* $(b \rightsquigarrow c) \to (a \rightsquigarrow c), (c \rightsquigarrow a) \to (c \rightsquigarrow b) \in F$*;*
(5) *filters* F *of* B *such that* $a \to b, b \rightsquigarrow c, c \to d \in F$ *imply* $a \rightsquigarrow d \in F$*.*

Proof. (1)⇒(2) This is obvious.

(2)⇒(1) Suppose $a\theta b$ and $c\theta d$. By (2), $(a \rightsquigarrow c)\theta(b \rightsquigarrow c)$ and $(b \rightsquigarrow c)\theta(b \rightsquigarrow d)$. Therefore, $(a \rightsquigarrow c)\theta(b \rightsquigarrow d)$. Thus, θ is a congruence of $(B, \rightsquigarrow)$.

(3)⇒(4) Suppose F satisfies (3) and $a \to b \in F$. Then $\Box(a \to b) \in F$. By (I12) and (I13), for any $c \in B$, we have $\Box(a \to b) \le (b \rightsquigarrow c) \to (a \rightsquigarrow c)$ and $\Box(a \to b) \le (c \rightsquigarrow a) \to (c \rightsquigarrow b)$. Therefore, $(b \rightsquigarrow c) \to (a \rightsquigarrow c), (c \rightsquigarrow a) \to (c \rightsquigarrow b) \in F$, and so F satisfies (4).

(4)⇒(5) Suppose F satisfies (4) and $a \to b, b \rightsquigarrow c, c \to d \in F$. From $a \to b \in F$ it follows that $(b \rightsquigarrow c) \to (a \rightsquigarrow c) \in F$. Therefore, since $b \rightsquigarrow c \in F$, we have $a \rightsquigarrow c \in F$. Also, from $c \to d \in F$ it follows that $(a \rightsquigarrow c) \to (a \rightsquigarrow d) \in F$. This together with $a \rightsquigarrow c \in F$ yields $a \rightsquigarrow d \in F$. Thus, F satisfies (5).

(5)⇒(3) Suppose F satisfies (5) and $a \in F$. Since $1 \to 1 = 1 \rightsquigarrow 1 = 1$ and $1 \to a = a$, we have $1 \to 1, 1 \rightsquigarrow 1, 1 \to a \in F$. Therefore, by (5), $\Box a = 1 \rightsquigarrow a \in F$. Thus, F satisfies (3).

(2)⇒(3) Suppose θ is a congruence of B and $a \in F_\theta$. Then $a\theta 1$. Therefore, $(1 \rightsquigarrow a)\theta(1 \rightsquigarrow 1)$. Thus, $\Box a\theta 1$, and so $\Box a \in F_\theta$.

(4)⇒(2) Suppose F satisfies (4), $a\theta_F b$, and $c \in B$. Then $a \to b \in F$ and $b \to a \in F$. Therefore, by (4), $(b \rightsquigarrow c) \to (a \rightsquigarrow c), (c \rightsquigarrow a) \to (c \rightsquigarrow b) \in F$ and $(a \rightsquigarrow c) \to (b \rightsquigarrow c), (c \rightsquigarrow b) \to (c \rightsquigarrow a) \in F$. Thus, $(a \rightsquigarrow c) \leftrightarrow (b \rightsquigarrow c), (c \rightsquigarrow a) \leftrightarrow (c \rightsquigarrow b) \in F$. Consequently, $(a \rightsquigarrow c)\theta_F(b \rightsquigarrow c)$ and $(c \rightsquigarrow a)\theta_F(c \rightsquigarrow b)$, and hence θ_F satisfies (2). □

Definition 4.8. Let $(B, \rightsquigarrow)$ be a strict implication algebra. We call a filter F of B a $\Box$-*filter* provided F satisfies Proposition 4.7(3); that is, $a \in F$ implies $\Box a \in F$.

By Proposition 4.7, congruences of strict implication algebras correspond to their $\Box$-filters. This is a generalization of a similar characterization of congruences of modal algebras (see, e.g., [10, Sec. 7.7]). For a strict implication algebra $(B, \rightsquigarrow)$ and $a \in B$, we use the usual abbreviation

$$\uparrow a := \{b \in B \mid a \le b\}.$$

Lemma 4.9. *Let* $(B, \rightsquigarrow)$ *be a strict implication algebra,* $a \in B$*, and* F *a* $\Box$*-filter. Then the filter generated by* $F \cup \{\Box a\}$ *is a* $\Box$*-filter. In particular, we have that* $\uparrow\Box a$ *and* $\uparrow\neg\Box a$ *are* $\Box$*-filters.*

Proof. Let F' be the filter generated by $F \cup \{\Box a\}$, and let $b \in F'$. Then there is $c \in F$ such that $c \wedge \Box a \le b$. As $\Box$ is an $\mathsf{S5}$-operator, we have $\Box(c \wedge \Box a) \le \Box b$. Since F is a $\Box$-filter, $\Box c \in F$. Therefore, again using the fact that $\Box$ is an $\mathsf{S5}$-operator, we obtain $\Box(c \wedge \Box a) = \Box c \wedge \Box\Box a = \Box c \wedge \Box a \in F'$. Thus, $\Box b \in F'$, which shows that F' is a $\Box$-filter.

In particular, as $\{1\}$ is a $\Box$-filter, it follows that $\uparrow\Box a$ is a $\Box$-filter, and by (I9) the same holds for $\uparrow\neg\Box a$. □

For a strict implication algebra $(B, \rightsquigarrow)$ and a $\Box$-filter F, let $(B/F, \rightsquigarrow_F)$ be the quotient algebra. For $a \in B$, we let $[a]$ be the corresponding element of B/F.

Lemma 4.10. *Let* $(B, \rightsquigarrow) \in \mathsf{SIA}$*.*

(1) *For a proper* $\Box$*-filter* F *in* $(B, \rightsquigarrow)$*, the following are equivalent:*
   (a) F *is a maximal proper* $\Box$*-filter.*
   (b) *For each* $a \in B$*, we have* $\Box a \in F$ *or* $\neg\Box a \in F$*.*
   (c) $(B/F, \rightsquigarrow_F) \in \mathsf{RSub}$*.*
(2) *If* F *is a* $\Box$*-filter and* $a \notin F$*, then there is a maximal* $\Box$*-filter* M *such that* $F \subseteq M$ *and* $a \notin M$*.*

Proof. (1) (a)⇒(b) Suppose $\Box a \notin F$. Let G be the filter generated by F and $\Box a$. By Lemma 4.9, G is a $\Box$-filter. Since F is a maximal $\Box$-filter, G is improper. Therefore, $0 = \Box a \wedge b$ for some $b \in F$. Thus, $b \leq \neg\Box a$, and so $\neg\Box a \in F$.

(b)⇒(c) Let $a \in B$. Then $\Box a \in F$ or $\neg\Box a \in F$. If $\Box a \in F$, then $\Box_F[a] = [\Box a] = 1_F$, where $\Box_F[a] = 1_F \rightsquigarrow_F [a]$. On the other hand, if $\Box a \notin F$, then $\neg\Box a \in F$, so $\neg_F\Box_F[a] = [\neg\Box a] = 1_F$, and hence $\Box_F[a] = 0_F$. This implies that $\{1_F\}$ and B/F are the only two $\Box_F$-filters in $(B/F, \rightsquigarrow_F)$. Thus, $(B/F, \rightsquigarrow_F)$ is a simple algebra, and hence $(B/F, \rightsquigarrow_F) \in \mathsf{RSub}$ by Proposition 3.3(2).

(c)⇒(a) Suppose G is a $\Box$-filter properly containing F. Then there is $a \in G \setminus F$. Since G is a $\Box$-filter and $\Box a \leq a$, we see that $\Box a \in G \setminus F$. Therefore, $[\Box a] \neq 1_F$. Since $(B/F, \rightsquigarrow_F) \in \mathsf{RSub}$, we conclude that $[\Box a] = 0_F$. Thus, $[\neg\Box a] = 1_F$, yielding that $\neg\Box a \in F \subseteq G$. Consequently, G is an improper $\Box$-filter, and hence F is a maximal $\Box$-filter.

(2) Since $a \notin F$, by Zorn's lemma there is a $\Box$-filter M such that $F \subseteq M$, $a \notin M$, and M is maximal with this property. If M is not a maximal $\Box$-filter, then by (1), there is $b \in B$ such that $\Box b, \neg\Box b \notin M$. Let G be the filter generated by M and $\Box b$ and H the filter generated by M and $\neg\Box b$. By Lemma 4.9, both G and H are $\Box$-filters that properly extend M. Therefore, $a \in G, H$, so there exist $c, d \in M$ such that $a \geq \Box b \wedge c$ and $a \geq \neg\Box b \wedge d$. Thus, $a \geq (\Box b \wedge c) \vee (\neg\Box b \wedge d) = (\Box b \vee \neg\Box b) \wedge (\Box b \vee d) \wedge (c \vee \neg\Box b) \wedge (c \vee d) \in M$. The obtained contradiction proves that M is a maximal $\Box$-filter. □

Theorem 4.11. *For a set of formulas Γ and a formula φ, we have:*

$$\Gamma \vdash_{\mathsf{SIC}} \varphi \quad \Leftrightarrow \quad \Gamma \models_{\mathsf{SIA}} \varphi \quad \Leftrightarrow \quad \Gamma \models_{\mathsf{RSub}} \varphi.$$

Proof. We already observed in Proposition 4.5 that $\Gamma \vdash_{\mathsf{SIC}} \varphi \Leftrightarrow \Gamma \models_{\mathsf{SIA}} \varphi$. This together with $\mathsf{RSub} \subseteq \mathsf{SIA}$ yields that $\Gamma \vdash_{\mathsf{SIC}} \varphi$ implies $\Gamma \models_{\mathsf{RSub}} \varphi$. Conversely, if $\Gamma \nvdash_{\mathsf{SIC}} \varphi$, then in the Lindenbaum algebra $(B, \rightsquigarrow)$ of SIC, the $\Box$-filter generated by $\{[\psi] \mid \psi \in \Gamma\}$ does not contain $[\varphi]$. By Lemma 4.10(2), there is a maximal $\Box$-filter F such that $\{[\psi] \mid \psi \in \Gamma\} \subseteq F$ and $[\varphi] \notin F$. But then $(B/F, \rightsquigarrow_F)$ satisfies Γ and refutes φ. By Lemma 4.10(1), $(B/F, \rightsquigarrow_F) \in \mathsf{RSub}$. Thus, $\Gamma \not\models_{\mathsf{RSub}} \varphi$. □

5. The symmetric strict implication calculus and its topological completeness

In this section we define the symmetric strict implication calculus $\mathsf{S^2IC}$ obtained by adding (A5) to SIC, and the corresponding variety $\mathsf{S^2IA}$ of symmetric strict implication algebras. We prove that $\mathsf{S^2IC}$ is strongly sound and complete with respect to Con, as well as with respect to Com and DeV. The last completeness together with de Vries duality allows us to introduce topological models for $\mathsf{S^2IC}$ based on compact Hausdorff spaces, and prove that $\mathsf{S^2IC}$ is strongly sound and complete with respect to the class of compact Hausdorff spaces.

Definition 5.1.

(1) We call a strict implication algebra $(B, \rightsquigarrow)$ *symmetric* if it satisfies (I5). Let $\mathsf{S^2IA}$ be the variety of symmetric strict implication algebras.
(2) The *symmetric strict implication calculus* $\mathsf{S^2IC}$ is obtained from the strict implication calculus SIC by postulating (A5).

Since (A5) corresponds to (I5), it follows from Proposition 4.5 that $\mathsf{S^2IC}$ is strongly sound and complete with respect to $\mathsf{S^2IA}$. Moreover, since for $(B, \rightsquigarrow) \in \mathsf{RSub}$ we have $(B, \rightsquigarrow) \in \mathsf{Con}$ iff $(B, \rightsquigarrow)$ satisfies (I5), the following is an immediate consequence of Theorem 4.11.

Theorem 5.2. *For a set of formulas* Γ *and a formula* φ*, we have:*

$$\Gamma \vdash_{\mathsf{S^2IC}} \varphi \quad \Leftrightarrow \quad \Gamma \models_{\mathsf{S^2IA}} \varphi \quad \Leftrightarrow \quad \Gamma \models_{\mathsf{Con}} \varphi.$$

We next show that each contact algebra can be embedded into a compingent algebra. For this we utilize the representation of subordination algebras discussed in Section 2, as well as the following result (cf. Lemma 2.2).

Lemma 5.3 ([13]). *Let* R *be a binary relation on a set* X*. Define* $\prec_R$ *on* $\mathcal{P}(X)$ *by* $U \prec_R V$ *iff* $R[U] \subseteq V$*.*

(1) $\prec_R$ *is a subordination on* $\mathcal{P}(X)$*.*
(2) R *is reflexive iff* $(\mathcal{P}(X), \prec_R)$ *satisfies* (S5)*.*
(3) R *is symmetric iff* $(\mathcal{P}(X), \prec_R)$ *satisfies* (S6)*.*
(4) R *is transitive iff* $(\mathcal{P}(X), \prec_R)$ *satisfies* (S7)*.*

We use Lemma 5.3 to show an analogue of [1, Lem. 2.5] in our setting. Let $(B, \prec)$ and $(C, \prec)$ be in RSub. We say that $(B, \prec)$ is *embedded* into $(C, \prec)$ if there is a Boolean embedding $h : B \to C$ such that $a \prec b$ iff $h(a) \prec h(b)$ for each $a, b \in B$.

Lemma 5.4.

(1) *Every* $(B, \prec) \in \mathsf{RSub}$ *can be embedded into some* $(C, \prec) \in \mathsf{RSub}$ *satisfying* (S7)*.*
(2) *Every* $(B, \prec) \in \mathsf{Con}$ *can be embedded into some* $(C, \prec) \in \mathsf{Con}$ *satisfying* (S7)*.*

Proof. (1) Suppose that (X, R) is the dual of $(B, \prec)$. By Lemma 2.2(1), R is reflexive. Let $Y = \{\{x, y\} \subseteq X \mid xRy\}$ and let

$$X' = \{(x, \alpha) \in X \times Y \mid x \in \alpha\}.$$

Define R' on X' by

$$(x, \alpha)R'(y, \beta) \Leftrightarrow xRy \text{ and } \alpha = \beta.$$

We show that R' is reflexive and transitive. That R' is reflexive follows from the reflexivity of R. To see that R' is transitive, let $(x, \alpha)R'(y, \beta)R'(z, \gamma)$. Then $xRyRz$ and $\alpha = \beta = \gamma$. Therefore, either $x = y$, $y = z$, or $z = x$. Since R is reflexive, we see that in each of these cases we have xRz. Thus, $(x, \alpha)R'(z, \gamma)$, and so R' is transitive.

Define $f : X' \to X$ by $f(x, \alpha) = x$. Clearly f is onto. Therefore, $f^{-1} : \mathsf{Clop}(X) \to \mathcal{P}(X')$ is a Boolean embedding.

Claim. For $U, V \in \mathsf{Clop}(X)$, we have $U \prec_R V$ iff $f^{-1}(U) \prec_{R'} f^{-1}(V)$.

Proof of claim. It follows from the definition of R' that $(x, \alpha)R'(y, \beta)$ implies $f(x, \alpha)Rf(y, \beta)$. So $U \prec_R V$ implies $f^{-1}(U) \prec_{R'} f^{-1}(V)$. For the converse, suppose $U \not\prec_R V$. Then $R[U] \not\subseteq V$. Therefore, there are $x \in U$ and $y \notin V$ such that xRy. Let $\alpha = \{x, y\}$. Then $(x, \alpha)R'(y, \alpha)$, $(x, \alpha) \in f^{-1}(U)$, and $(y, \alpha) \notin f^{-1}(V)$. Thus, $R'[f^{-1}(U)] \not\subseteq f^{-1}(V)$, and hence $f^{-1}(U) \not\prec_{R'} f^{-1}(V)$. □

Let $(C, \prec) = (\mathcal{P}(X'), \prec_{R'})$. By Lemma 5.3, $(C, \prec)$ satisfies (S1)–(S5) and (S7), and by the Claim, f^{-1} is an embedding of $(B, \prec)$ into $(C, \prec)$.

(2) If $(B, \prec) \in \mathsf{Con}$, then by Lemma 2.2(2), R is also symmetric. Therefore, so is R', and hence R' is an equivalence relation. Thus, by Lemma 5.3, $(C, \prec)$ satisfies (S1)–(S7), concluding the proof. □

Lemma 5.5. *Suppose* $(B, \prec) \in \mathsf{RSub}$*. Let* $B' = B \times B$ *and define* $\prec'$ *on* B' *by*

$$(a, b) \prec' (c, d) \Leftrightarrow a \prec c \text{ and } b \leq d.$$

Then $(B', \prec') \in \mathsf{RSub}$*. Moreover, if* $(B, \prec) \in \mathsf{Con}$*, then* $(B', \prec') \in \mathsf{Con}$*.*

Proof. Since $(B, \prec) \in \mathsf{RSub}$, it satisfies (S1)–(S5). We show that $(B', \prec')$ also satisfies (S1)–(S5).

(S1) Since $0 \prec 0$ and $1 \prec 1$, it is obvious that $(0, 0) \prec' (0, 0)$ and $(1, 1) \prec' (1, 1)$.

(S2) Suppose $(a, b) \prec (c, d), (c', d')$. Then $a \prec c, c'$ and $b \leq d, d'$. Therefore, $a \prec c \wedge c'$ and $b \leq d \wedge d'$. Thus, $(a, b) \prec' (c, c') \wedge (d, d')$.

(S3) Suppose $(a, b), (a', b') \prec' (c, d)$. Then $a, a' \prec c$ and $b, b' \leq d$. Therefore, $a \vee a' \prec c$ and $b \vee b' \leq d$. Thus, $(a, b) \vee (a', b') \prec' (c, d)$.

(S4) Suppose $(a, b) \leq (a', b') \prec (c', d') \leq (c, d)$. Then $a \leq a' \prec c' \leq c$ and $b \leq b' \leq d' \leq d$. Thus, $a \prec c$ and $b \leq d$, and so $(a, b) \prec' (c, d)$.

(S5) Suppose $(a, b) \prec' (c, d)$. Then $a \prec c$ and $b \leq d$. From $a \prec c$ it follows that $a \leq c$. Thus, $(a, b) \leq (c, d)$.

Now suppose that in addition $(B, \prec) \in \mathsf{Con}$. Then $(B, \prec)$ satisfies (S6). We show that $(B', \prec')$ also satisfies (S6).

(S6) Suppose $(a, b) \prec' (c, d)$. Then $a \prec c$ and $b \leq d$. Therefore, $\neg c \prec \neg a$ and $\neg d \leq \neg b$. Thus, $\neg(c, d) \prec' \neg(a, b)$.

Consequently, if $(B, \prec) \in \mathsf{Con}$, then $(B', \prec') \in \mathsf{Con}$. □

Lemma 5.6.

(1) *Every* $(B, \prec) \in \mathsf{RSub}$ *can be embedded into* $(C, \prec) \in \mathsf{RSub}$ *satisfying* (S8)*.*
(2) *In addition, if* $(B, \prec)$ *satisfies either* (S6) *or* (S7)*, then so does* $(C, \prec)$*.*

Proof. (1) Starting from $(B, \prec)$, we inductively build a chain

$$(B, \prec) \hookrightarrow (B_1, \prec) \hookrightarrow (B_2, \prec) \hookrightarrow (B_3, \prec) \hookrightarrow \cdots$$

in RSub such that the union $(C, \prec) := \bigcup_{n \in \omega} (B_n, \prec)$ satisfies (S8).

If $(B_n, \prec)$ is already defined, define $(B_{n+1}, \prec) := (B_n, \prec) \times (B_n, \leq)$. By Lemma 5.5, $(B_{n+1}, \prec) \in \mathsf{RSub}$. Moreover, $a \mapsto (a, a)$ is an embedding of $(B_n, \prec)$ into $(B_{n+1}, \prec)$. We prove that $(C, \prec)$ satisfies (S8).

Let $0 \neq a \in C$. Then there is n such that $a \in B_n$. Therefore, $(a, a) \in B_{n+1}$. Let $b := (0, a) \in B_{n+1}$. We have $b \neq 0$ and $b \prec (a, a)$. Thus, $(C, \prec)$ satisfies (S8).

(2) If $(B, \prec)$ satisfies (S6), then each $(B_n, \prec)$ satisfies (S6) by Lemma 5.5. Therefore, so does $(C, \prec)$.

Finally, suppose that in addition $(B, \prec)$ satisfies (S7). We show that if $(B_n, \prec)$ satisfies (S7), then so does $(B_{n+1}, \prec)$. Let $(a_1, a_2) \prec (b_1, b_2)$ in $(B_{n+1}, \prec)$. Then $a_1 \prec b_1$ and $a_2 \leq b_2$ in B_n. By (S7), there exists $c \in B_n$ such that $a_1 \prec c$ and $c \prec b_1$. So, for $(c, a_2) \in B_{n+1}$, we have $(a_1, a_2) \prec (c, a_2) \prec (b_1, b_2)$. Therefore, $(B_{n+1}, \prec)$ satisfies (S7). Thus, by induction, each $(B_n, \prec)$ satisfies (S7). By the Chang-Łoś-Suszko theorem (see, e.g., [11, Thm. 3.2.3]), Π_2-sentences are preserved by direct limits. Since (S7) is a Π_2-sentence, the direct limit $(C, \prec)$ of the chain also satisfies (S7). □

Let $(B, \rightsquigarrow)$ and $(C, \rightsquigarrow)$ be the strict implication algebras corresponding to $(B, \prec)$ and $(C, \prec)$, respectively. It is straightforward to check that $h : B \to C$ is an embedding of $(B, \prec)$ into $(C, \prec)$ iff h is an isomorphism from $(B, \rightsquigarrow)$ to a subalgebra of $(C, \rightsquigarrow)$. For a class $\mathcal{K}$ of strict implication algebras, let $\mathbf{IS}(\mathcal{K})$ be the class of isomorphic copies of subalgebras of algebras in $\mathcal{K}$.

Theorem 5.7. $\mathbf{IS}(\mathsf{Com}) = \mathsf{Con}$.

Proof. Obviously, $\mathsf{Com} \subseteq \mathsf{Con}$ and as Con is a universal class, we have that $\mathbf{IS}(\mathsf{Com}) \subseteq \mathsf{Con}$. Conversely, suppose $(B, \rightsquigarrow) \in \mathsf{Con}$. Then by Lemmas 5.4(2) and 5.6 it is isomorphic to a subalgebra of $(C, \rightsquigarrow) \in \mathsf{Com}$. Therefore, $\mathsf{Con} \subseteq \mathbf{IS}(\mathsf{Com})$. □

Theorem 5.8. $\mathsf{S^2IC}$ *is strongly sound and complete with respect to* Com *i.e., for a set of formulas* Γ *and a formula* φ*, we have:*

$$\Gamma \vdash_{\mathsf{S^2IC}} \varphi \quad \Leftrightarrow \quad \Gamma \models_{\mathsf{Com}} \varphi.$$

Proof. The left to right direction follows from Theorem 5.2 and the fact that $\mathsf{Com} \subseteq \mathsf{Con}$. Now suppose $\Gamma \nvdash_{\mathsf{S^2IC}}$. Applying Theorem 5.2 again yields a contact algebra $(B, \rightsquigarrow)$ and a valuation v on B such that $v(\gamma) = 1_B$ for each $\gamma \in \Gamma$ and $v(\varphi) \neq 1_B$. By Theorem 5.7, there is $(C, \rightsquigarrow) \in \mathsf{Com}$ such that $(B, \rightsquigarrow)$ is isomorphic to a subalgebra of $(C, \rightsquigarrow)$. We may view v as a valuation on C, so $v(\gamma) = 1_C$ for each $\gamma \in \Gamma$ and $v(\varphi) \neq 1_C$. Thus, $\Gamma \not\models_{\mathsf{Com}} \varphi$. □

We recall that the de Vries algebra of a compact Hausdorff space X is the pair $(\mathcal{RO}(X), \prec)$, where $\mathcal{RO}(X)$ is the complete Boolean algebra of regular open subsets of X and $U \prec V$ iff $\mathsf{Cl}(U) \subseteq V$. By de Vries duality [12], every de Vries algebra is isomorphic to the de Vries algebra of some compact Hausdorff space. This allows us to define topological semantics for our language.

Definition 5.9. A *compact Hausdorff model* is a pair (X, v), where X is a compact Hausdorff space and v is a valuation assigning a regular open set to each propositional letter.

If $\rightsquigarrow$ is the strict implication corresponding to $\prec$, then the formulas of our language are interpreted in $(\mathcal{RO}(X), \rightsquigarrow) \in \mathsf{DeV}$.

Theorem 5.10.

(1) *The system* $\mathsf{S^2IC}$ *is strongly sound and complete with respect to* DeV.
(2) *The system* $\mathsf{S^2IC}$ *is strongly sound and complete with respect to compact Hausdorff models.*

Proof. (1) Since $\mathsf{DeV} \subseteq \mathsf{Com}$, by Theorem 5.8, we have $\Gamma \vdash_{\mathsf{S^2IC}} \varphi$ implies $\Gamma \models_{\mathsf{DeV}} \varphi$. Conversely, suppose $\Gamma \nvdash_{\mathsf{S^2IC}} \varphi$. Applying Theorem 5.8 again yields a compingent algebra $(B, \rightsquigarrow)$ and a valuation v on B such that $v(\gamma) = 1_B$ for each $\gamma \in \Gamma$ and $v(\varphi) \neq 1_B$. By de Vries duality, there is a compact Hausdorff space X such that $(B, \rightsquigarrow)$ embeds into $(\mathcal{RO}(X), \rightsquigarrow)$. We may view v as a valuation on $\mathcal{RO}(X)$, so $v(\gamma) = X$ for each $\gamma \in \Gamma$ and $v(\varphi) \neq X$. Since $(\mathcal{RO}(X), \rightsquigarrow) \in \mathsf{DeV}$, we conclude that $\Gamma \not\models_{\mathsf{DeV}} \varphi$.

(2) This follows from (1) and de Vries duality. □

Remark 5.11. Let $(B, \prec) \in \mathsf{Com}$ and let $\beta : B \to \mathcal{RO}(X)$ be the embedding. By [12, Thm. I.3.9], for $U, V \in \mathcal{RO}(X)$ with $U \prec V$, there are $a, b \in B$ with $a \prec b$, $U \subseteq \beta(a)$, and $\beta(b) \subseteq V$. From this it follows that $\mathcal{RO}(X)$ is isomorphic to the MacNeille completion of B. Thus, it is possible to prove Theorem 5.10(1) without using the de Vries representation of compingent algebras. Namely, for $(B, \prec) \in \mathsf{Com}$, let $\overline{B}$ be the MacNeille completion of B. By identifying B with its image, we may view B as a subalgebra of $\overline{B}$, and define $\lhd$ on $\overline{B}$ by setting

$$x \lhd y \text{ iff there exist } a, b \in B \text{ such that } x \leq a \prec b \leq y.$$

A direct verification shows that $(\overline{B}, \lhd) \in \mathsf{DeV}$, which yields a point-free proof of Theorem 5.10(1); see [5, Lem. 6.5] for details.

6. Π_2-rules, admissibility, and further completeness results

6.1. Π_2-rules

As we saw in the previous section, $\mathsf{S^2IC}$ is strongly sound and complete with respect to DeV, and hence $\mathsf{S^2IC}$ is the strict implication logic of compact Hausdorff models. Note that neither (I6) nor (I7) is expressible in our logic as $\mathsf{S^2IC}$ is also strongly sound and complete with respect to Con. This generates an interesting question of what logical formalism to use when reasining about compact Hausdorff models (see Remark 6.23 for more discussion). In this section we show that we can express (I6) and (I7) in our propositional language by means of Π_2-rules. For this we first rewrite (I6) and (I7) in the following form.

(Π6) $\forall x_1, x_2, y\Big(x_1 \rightsquigarrow x_2 \not\leq y \to \exists z : (x_1 \rightsquigarrow z) \wedge (z \rightsquigarrow x_2) \not\leq y\Big)$;

(Π7) $\forall x, y\Big(x \not\leq y \to \exists z : z \wedge (z \rightsquigarrow x) \not\leq y\Big)$

Lemma 6.1. *Let* $(B, \rightsquigarrow) \in \mathsf{RSub}$.

(1) $(B, \rightsquigarrow) \models$ (I6) *iff* $(B, \rightsquigarrow) \models$ (Π6).
(2) $(B, \rightsquigarrow) \models$ (I7) *iff* $(B, \rightsquigarrow) \models$ (Π7).

Proof. (1) ($\Rightarrow$) Suppose $(B, \rightsquigarrow) \models$ (I6). Let $a, b, d \in B$ be such that $a \rightsquigarrow b \not\leq d$. Then $d \neq 1$ and $a \rightsquigarrow b \neq 0$, so $a \rightsquigarrow b = 1$ since $(B, \rightsquigarrow) \in \mathsf{RSub}$. By (I6), there is $c \in B$ such that $a \rightsquigarrow c = c \rightsquigarrow b = 1$. Therefore, $1 = (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) \not\leq d$. Thus, $(B, \rightsquigarrow) \models$ (Π6).

($\Leftarrow$) Suppose $(B, \rightsquigarrow) \models$ (Π6). Let $a, b \in B$ be such that $a \rightsquigarrow b = 1$. Then $a \rightsquigarrow b \not\leq 0$. By ($\Pi$6), there is $c \in B$ such that $(a \rightsquigarrow c) \wedge (c \rightsquigarrow b) \not\leq 0$. Therefore, since $(B, \rightsquigarrow) \in \mathsf{RSub}$, we have $a \rightsquigarrow c = c \rightsquigarrow b = 1$. Thus, $(B, \rightsquigarrow) \models$ (I6).

(2) ($\Rightarrow$) Suppose $(B, \rightsquigarrow) \models$ (I7). Let $a, c \in B$ be such that $a \not\leq c$. Then $a \wedge \neg c \neq 0$. By (I7), there is $b \neq 0$ such that $b \rightsquigarrow (a \wedge \neg c) = 1$. By (I3), $b \rightsquigarrow (a \wedge \neg c) = (b \rightsquigarrow a) \wedge (b \rightsquigarrow \neg c)$. Therefore, $b \rightsquigarrow a = 1$ and $b \rightsquigarrow \neg c = 1$. The latter equality, by (I4), yields $b \leq \neg c$. Since $b \neq 0$, we must have $b \not\leq c$. Thus, we have found $b \in B$ such that $b \wedge (b \rightsquigarrow a) = b \not\leq c$. This shows that $(B, \rightsquigarrow) \models$ (Π7).

($\Leftarrow$) Suppose $(B, \rightsquigarrow) \models$ (Π7). Let $a \neq 0$ be an element of B. By (Π7), there is $b \in B$ such that $b \wedge (b \rightsquigarrow a) \not\leq 0$. Therefore, $b \neq 0$ and $b \rightsquigarrow a = 1$. Thus, $(B, \rightsquigarrow) \models$ (I7). □

We next show that $\forall\exists$-statements can be expressed by means of non-standard rules, which we call Π_2-rules. The use of non-standard rules in modal logic is not new. One of the pioneers of this approach was Gabbay [16], who introduced a non-standard rule for irreflexivity. A precursor to this work was Burgess [8] who used such rules in the context of branching time logic. We also refer to [17] for the application of non-standard rules to axiomatize the logic of the real line in the language with the Since and Until modalities, and to [28] for a general completeness result for modal languages that are sufficiently expressive to define the so-called difference modality. Our approach is closest to that of Balbiani et al. [1] who use similar non-standard rules in the context of region-based theories of space.

Definition 6.2 (Π_2-rule). A Π_2-*rule* is a rule of the form

$$(\rho) \qquad \frac{F(\overline{\varphi}, \overline{p}) \to \chi}{G(\overline{\varphi}) \to \chi}$$

where F, G are formulas, $\overline{\varphi}$ is a tuple of formulas, χ is a formula, and $\overline{p}$ is a tuple of propositional letters which do not occur in $\overline{\varphi}$ and χ.

To each Π_2-rule ρ, we associate the $\forall\exists$-statement

$$\Pi(\rho) := \forall \overline{x}, z\Big(G(\overline{x}) \not\leq z \to \exists \overline{y} : F(\overline{x}, \overline{y}) \not\leq z\Big).$$

Definition 6.3. We say that a strict implication algebra $(B, \rightsquigarrow)$ *validates* a Π_2-rule ρ, and write $(B, \rightsquigarrow) \models \rho$, provided $(B, \rightsquigarrow)$ satisfies $\Pi(\rho)$.

Consider the Π_2-rules:

(ρ6) $\dfrac{(\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$;

(ρ7) $\dfrac{p \wedge (p \rightsquigarrow \varphi) \to \chi}{\varphi \to \chi}$

It is easy to see that $\Pi(\rho6) = (\Pi6)$ and $\Pi(\rho7) = (\Pi7)$, so by Lemma 6.1, for each $(B, \rightsquigarrow) \in \mathsf{RSub}$, we have:

$$(B, \rightsquigarrow) \models (\rho6) \text{ iff } (B, \rightsquigarrow) \models (\text{I6});$$
$$(B, \rightsquigarrow) \models (\rho7) \text{ iff } (B, \rightsquigarrow) \models (\text{I7}).$$

Definition 6.4 (Proofs with Π_2-rules). Let Σ be a set of Π_2-rules. For a set of formulas Γ and a formula φ, we say that φ is *derivable* from Γ in SIC using the Π_2-rules in Σ, and write $\Gamma \vdash_\Sigma \varphi$, provided there is a proof $\psi_1, \dots, \psi_n$ such that $\psi_n = \varphi$ and each ψ_i is in Γ, or is an instance of an axiom of SIC, or is obtained either by (MP) or (N) from some previous ψ_j's, or there is $j < i$ such that ψ_i is obtained from ψ_j by an application of one of the Π_2-rules $\rho \in \Sigma$; that is, $\psi_j = F(\overline{\xi}, \overline{p}) \to \chi$ and $\psi_i = G(\overline{\xi}) \to \chi$, where F, G are formulas, $\overline{\xi}$ is a tuple of formulas, χ is a formula, and $\overline{p}$ is a tuple of propositional letters not occurring in $\overline{\xi}, \chi$ or any of the formulas from Γ that are used in $\psi_1, \dots, \psi_{i-1}$ as assumptions.

We next show that the deduction theorem remains true when proofs also involve Π_2-rules.

Lemma 6.5. *For any set Γ of formulas and for any formulas φ, ψ, we have:*

$$\Gamma \cup \{\varphi\} \vdash_\Sigma \psi \quad \Leftrightarrow \quad \Gamma \vdash_\Sigma \Box\varphi \to \psi.$$

Proof. ($\Leftarrow$) Same as the corresponding proof of Lemma 4.4.

($\Rightarrow$) The only step that is not covered in the corresponding proof of Lemma 4.4 is the step of applying some Π_2-rule $\rho \in \Sigma$: Suppose there is $j < i$ such that $\psi_j = F(\overline{\xi}, \overline{p}) \to \chi$ and $\psi_i = G(\overline{\xi}) \to \chi$ for some formulas F, G, a tuple of formulas $\overline{\xi}$, a formula χ, and a tuple $\overline{p}$ of fresh propositional letters not occurring in any of the formulas involved in the proof. We may assume without loss of generality that $\overline{p}$ also does not occur in φ. If this is not the case, then we can rewrite a proof of ψ_i from $\Gamma \cup \{\varphi\}$ so that the propositional letters $\overline{p}$ are replaced with fresh propositional letters. By inductive hypothesis, $\Gamma \vdash_\Sigma \Box\varphi \to (F(\overline{\xi}, \overline{p}) \to \chi)$. Thus, $\Gamma \vdash_\Sigma F(\overline{\xi}, \overline{p}) \to (\Box\varphi \to \chi)$. Applying ρ yields $\Gamma \vdash_\Sigma G(\overline{\xi}) \to (\Box\varphi \to \chi)$. From this we conclude that $\Gamma \vdash_\Sigma \Box\varphi \to (G(\overline{\xi}) \to \chi)$, as desired. $\Box$

Let $\mathcal{S}$ be the system obtained by adding the Π_2-rules $\{\rho_n \mid n \in \mathbb{N}\}$ to SIC. Let also $\mathcal{U}$ be the inductive subclass of RSub defined by the $\forall\exists$-statements $\{\Pi(\rho_n) \mid n \in \mathbb{N}\}$. We next show that $\mathcal{S}$ is strongly sound and complete with respect to $\mathcal{U}$. The proof is a modification of the standard Lindenbaum construction (see, e.g., [23]). The modification follows a similar pattern to the one given in [1, Lem. 7.10].

Theorem 6.6. *Let $\mathcal{S} = \mathsf{SIC} + \{\rho_n \mid n \in \mathbb{N}\}$, let $\mathcal{U}$ be the inductive subclass of RSub defined by $\{\Pi(\rho_n) \mid n \in \mathbb{N}\}$, and let $\mathcal{V}$ be the variety generated by $\mathcal{U}$. For a set of formulas Γ and a formula φ, we have:*

(1) $\Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow \Gamma \models_{\mathcal{U}} \varphi$.
(2) $\vdash_{\mathcal{S}} \varphi \Leftrightarrow \models_{\mathcal{V}} \varphi$.

Proof. (1) That $\Gamma \vdash_{\mathcal{S}} \varphi \Rightarrow \Gamma \models_{\mathcal{U}} \varphi$ is a straightforward inductive proof on the length of derivations. We only consider the case of Π_2-rules.

Suppose ρ is a Π_2-rule in $\mathcal{S}$ as defined in Definition 6.2, v is a valuation into $(B, \rightsquigarrow) \in \mathsf{SIA}$ satisfying $\Pi(\rho)$, and $v(\gamma) = 1$ for each $\gamma \in \Gamma$. For simplicity of notation we will write $v(\Gamma) = 1$ when $v(\gamma) = 1$ for each $\gamma \in \Gamma$. We may assume without loss of generality (by re-enumerating all the propositional letters if need be) that the propositional letters $\overline{p}$ do not occur in any of the formulas in Γ. If $G(v(\overline{\xi})) \nleq v(\chi)$, then since $(B, \rightsquigarrow)$ satisfies $\Pi(\rho)$, there is a tuple $\overline{c}$ in B such that $F(v(\overline{\xi}), \overline{c}) \nleq v(\chi)$. Consider the valuation v' which coincides with v everywhere, except maps $\overline{p}$ to $\overline{c}$. Then $v'(F(\overline{\xi}, \overline{p})) = F(v(\overline{\xi}), \overline{c}) \nleq v(\chi) = v'(\chi)$, so $v'(F(\overline{\xi}, \overline{p}) \to \chi) \neq 1$. Since v' coincides with v on all propositional letters except $\overline{p}$ and since we assumed that $\overline{p}$ do not occur in Γ, we have $v'(\Gamma) = v(\Gamma) = 1$. So we have found a valuation v' such that $v'(\Gamma) = 1$ and $v'(F(\overline{\xi}, \overline{p}) \to \chi) \neq 1$, contradicting the inductive hypothesis.

To complete the proof of (1), it remains to show that $\Gamma \nvdash_{\mathcal{S}} \varphi \Rightarrow \Gamma \not\models_{\mathcal{U}} \varphi$. We do this by slightly modifying the construction of the Lindenbaum algebra. Suppose $\Gamma \nvdash_{\mathcal{S}} \varphi$. For each rule ρ_i, we add a countably infinite set of fresh propositional letters to the set of existing propositional letters. Then we build the Lindenbaum algebra $(B, \rightsquigarrow)$ over the expanded set of propositional letters, where the elements are the equivalence classes $[\varphi]$ under provable equivalence in $\mathcal{S}$. Next we construct a maximal $\Box$-filter M of $(B, \rightsquigarrow)$ such that $\{[\psi] \mid \psi \in \Gamma\} \cup \{[\neg\Box\varphi]\} \subseteq M$ and for every rule ρ_i and formulas $\overline{\varphi}, \chi$:

(†) if $[G_i(\overline{\varphi}) \to \chi] \notin M$, then there is a tuple $\overline{p}$ such that $[F_i(\overline{\varphi}, \overline{p}) \to \chi] \notin M$.

To construct M, let $A_0 := \Gamma \cup \{\neg\Box\varphi\}$, a consistent set. We enumerate all formulas φ as $(\varphi_k : k \in \mathbb{N})$ and all tuples $(i, \overline{\varphi}, \chi)$ where $i \in \mathbb{N}$ and $\overline{\varphi}, \chi$ are as in the particular rule ρ_i, and we build the sets $A_0 \subseteq A_1 \subseteq \dots \subseteq A_n \subseteq \dots$ as follows:

- For $n = 2k$, if $A_n \nvdash_{\mathcal{S}} \Box\varphi_k$, let $A_{n+1} = A_n \cup \{\neg\Box\varphi_k\}$; otherwise let $A_{n+1} = A_n$.
- For $n = 2k+1$, let $(l, \overline{\varphi}, \chi)$ be the k-th tuple. If $A_n \nvdash_{\mathcal{S}} G_l(\overline{\varphi}) \to \chi$, let $A_{n+1} = A_n \cup \{\neg(F_l(\overline{\varphi}, \overline{p}) \to \chi)\}$, where $\overline{p}$ is a tuple of propositional letters for ρ_l not occurring in $\overline{\varphi}, \chi$, and any of ψ with $\psi \in A_n$ (we can take $\overline{p}$ from the countably infinite additional propositional letters which we have reserved for the rule ρ_l). Otherwise, let $A_{n+1} = A_n$.

Let $A = \bigcup_{n \in \mathbb{N}} A_n$, $S_A = \{\psi \mid A \vdash_{\mathcal{S}} \psi\}$, and $M = \{[\psi] \mid \psi \in S_A\}$. It is easy to see that $S_A \vdash_{\mathcal{S}} \varphi$ implies $\varphi \in S_A$, so M is a filter. Also, for any Γ we have $\Gamma \vdash_{\mathcal{S}} \varphi \Rightarrow \Gamma \vdash_{\mathcal{S}} \Box\varphi$. Therefore, M is a $\Box$-filter. Moreover, all A_n are consistent, and hence so is S_A. This implies that M is a proper $\Box$-filter. Thus, by the even steps of the construction of the sets A_n, and by Lemma 4.10(1), M is a maximal $\Box$-filter.

Because $A \subseteq S_A$, we have $\{[\psi] \mid \psi \in \Gamma\} \cup \{[\neg\Box\varphi]\} \subseteq M$. Finally, the odd steps of the construction of the sets A_n ensure that M satisfies (†). Therefore, we can conclude that M satisfies all the desired properties.

By (†), the quotient of $(B, \rightsquigarrow)$ by M satisfies each $\Pi(\rho_i)$. By Lemma 4.10(1), the quotient belongs to RSub. Therefore, the quotient belongs to $\mathcal{U}$. Moreover, since $[\neg\Box\varphi] \in M$, we have that $[\neg\Box\varphi]$ maps to 1, so $[\Box\varphi]$ maps to 0 in the quotient. Thus, $[\varphi]$ does not map to 1 in the quotient, while Γ does, and hence $\Gamma \not\models_{\mathcal{U}} \varphi$.

(2) Observe that $\mathcal{U}$ consists of the subdirectly irreducible members of $\mathcal{V}$, and apply (1). □

The class of subdirectly irreducible algebras in SIA validating a set of Π_2-rules is an elementary inductive subclass of RSub. We next show that the converse is also true. Namely, for every elementary inductive subclass $\mathcal{U}$ of RSub, there is a set of Π_2-rules $\{\rho_i \mid i \in I\}$ such that $\mathcal{S} = \mathsf{SIC} + \{\rho_i \mid i \in I\}$ is strongly sound and complete with respect to $\mathcal{U}$. To obtain such a set of Π_2-rules, it is sufficient to show that every

Π_2-statement is equivalent to a statement of the form $\Pi(\rho)$ for some Π_2-rule ρ. Without loss of generality we may assume that all atomic formulas $\Phi(\overline{x}, \overline{y})$ are of the form $t(\overline{x}, \overline{y}) = 1$ for some term t.

Definition 6.7. Given a quantifier-free first-order formula $\Phi(\overline{x}, \overline{y})$, we associate with the tuples of variables $\overline{x}, \overline{y}$ the tuples of propositional letters $\overline{p}, \overline{q}$, and define the formula $\Phi^*(\overline{p}, \overline{q})$ in the language of SIC as follows:

$$(t(\overline{x}, \overline{y}) = 1)^* = \Box t(\overline{p}, \overline{q})$$
$$(\neg\Psi)^*(\overline{x}, \overline{y}) = \neg\Psi^*(\overline{p}, \overline{q})$$
$$(\Psi_1(\overline{x}, \overline{y}) \wedge \Psi_2(\overline{x}, \overline{y}))^* = \Psi_1^*(\overline{p}, \overline{q}) \wedge \Psi_2^*(\overline{p}, \overline{q})$$

Lemma 6.8. *Let $(B, \rightsquigarrow) \in \mathsf{RSub}$ and $\Phi(\overline{x}, \overline{y})$ be a quantifier-free formula.*

(1) *$(B, \rightsquigarrow)$ satisfies $\Phi(\overline{x}, \overline{y})$ iff $(B, \rightsquigarrow)$ satisfies the formula $\Phi^*(\overline{p}, \overline{q})$.*
(2) *$(B, \rightsquigarrow)$ satisfies $\forall\overline{x}\exists\overline{y}\Phi(\overline{x}, \overline{y})$ iff $(B, \rightsquigarrow)$ satisfies $\forall\overline{x}, z\Big(1 \nleq z \to \exists\overline{y} : \Phi^*(\overline{x}, \overline{y}) \nleq z\Big)$.*

Proof. (1) For each term $t(\overline{x}, \overline{y})$, we evaluate $\overline{x}, \overline{p}$ as $\overline{a}$ and $\overline{y}, \overline{q}$ as $\overline{b}$, where $\overline{a}, \overline{b}$ are tuples of elements of B. It is obvious that $t(\overline{a}, \overline{b}) = 1$ implies $\Box t(\overline{a}, \overline{b}) = 1$, and $t(\overline{a}, \overline{b}) \neq 1$ implies $\Box t(\overline{a}, \overline{b}) = 0$. This shows the equivalence for atomic formulas, and an easy induction then proves it for all quantifier-free formulas.

(2) ($\Rightarrow$) Suppose $(B, \rightsquigarrow) \models \forall\overline{x}\exists\overline{y}\Phi(\overline{x}, \overline{y})$. Let $\overline{a}$ be a tuple of elements of B and $c \in B$. By assumption, there exists a tuple $\overline{b}$ in B such that $(B, \rightsquigarrow) \models \Phi(\overline{x}, \overline{y})[\overline{a}, \overline{b}]$. Therefore, by (1), if $1 \nleq c$, then $\Phi^*(\overline{a}, \overline{b}) = 1 \nleq c$. Thus, $(B, \rightsquigarrow) \models \forall\overline{x}, z\Big(1 \nleq z \to \exists\overline{y} : \Phi^*(\overline{x}, \overline{y}) \nleq z\Big)$.

($\Leftarrow$) Suppose $(B, \rightsquigarrow) \models \forall\overline{x}, z\Big(1 \nleq z \to \exists\overline{y} : \Phi^*(\overline{x}, \overline{y}) \nleq z\Big)$. Let $\overline{a}$ be a tuple of elements of B. Since $1 \nleq 0$, there exists a tuple $\overline{b}$ in B such that $\Phi^*(\overline{a}, \overline{b}) \nleq 0$. Therefore, since $\Phi^*(\overline{a}, \overline{b})$ evaluates only to 0 or 1, we obtain $\Phi^*(\overline{a}, \overline{b}) = 1$. Thus, by (1), $(B, \rightsquigarrow) \models \Phi(\overline{x}, \overline{y})[\overline{a}, \overline{b}]$. This shows that $(B, \rightsquigarrow) \models \forall\overline{x}\exists\overline{y}\Phi(\overline{x}, \overline{y})$. □

Consequently, an arbitrary Π_2-statement $\forall\overline{x}\exists\overline{y}\Phi(\overline{x}, \overline{y})$ is equivalent to the Π_2-statement associated to the Π_2-rule

$$(\rho_\Phi) \quad \frac{\Phi^*(\overline{\varphi}, \overline{p}) \to \chi}{\chi}$$

Thus, by Theorem 6.6, we obtain:

Theorem 6.9. *If T is a Π_2-theory of first-order logic axiomatizing an inductive subclass $\mathcal{U}$ of RSub, then the system $\mathcal{S} = \mathsf{SIC} + \{\rho_\Phi \mid \Phi \in T\}$ is strongly sound and complete with respect to $\mathcal{U}$; that is, for a set of formulas Γ and a formula φ, we have:*

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \Leftrightarrow \quad \Gamma \models_{\mathcal{U}} \varphi.$$

6.2. Admissibility of Π_2-rules

By Theorem 5.8, $\mathsf{S^2IC}$ is strongly sound and complete with respect to Com. On the other hand, it follows from Theorem 6.9 that $\mathsf{S^2IC}$ together with (ρ6) and (ρ7) is also strongly sound and complete with respect to Com. Therefore, the rules (ρ6) and (ρ7) are admissible in $\mathsf{S^2IC}$. We next give a general criterion of admissibility for Π_2-rules in SIC and $\mathsf{S^2IC}$. This yields an alternative proof that (ρ6) and (ρ7) are admissible in $\mathsf{S^2IC}$.

Definition 6.10. A rule ρ is *admissible* in a system $\mathcal{S}$ if for each formula φ, from $\vdash_{\mathcal{S}+\rho} \varphi$ it follows that $\vdash_{\mathcal{S}} \varphi$.

Lemma 6.11. *If a* Π_2*-rule*

$$(\rho) \quad \frac{F(\overline{\varphi}, \overline{p}) \to \chi}{G(\overline{\varphi}) \to \chi}$$

is admissible in $\mathcal{S} \supseteq \mathsf{SIC}$*, then* $\Gamma \vdash_{\mathcal{S}+\rho} \varphi \Leftrightarrow \Gamma \vdash_{\mathcal{S}} \varphi$*.*

Proof. It suffices to show that for any set of formulas Γ and any tuple $\overline{\varphi}, \chi$ of formulas, if $\Gamma \vdash_{\mathsf{SIC}} F(\overline{\varphi}, \overline{p}) \to \chi$ and $\overline{p}$ does not occur in $\Gamma, \overline{\varphi}, \chi$, then $\Gamma \vdash_{\mathsf{SIC}} G(\overline{\varphi}) \to \chi$.

Suppose $\Gamma \vdash_{\mathsf{SIC}} F(\overline{\varphi}, \overline{p}) \to \chi$ and $\overline{p}$ does not occur in $\Gamma, \overline{\varphi}, \chi$. Then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathsf{SIC}} F(\overline{\varphi}, \overline{p}) \to \chi$. Let $\psi = \bigwedge \Gamma_0$, so $\{\psi\} \vdash_{\mathsf{SIC}} F(\overline{\varphi}, \overline{p}) \to \chi$. By Theorem 4.4, $\vdash_{\mathsf{SIC}} \Box\psi \to (F(\overline{\varphi}, \overline{p}) \to \chi)$, so $\vdash_{\mathsf{SIC}} F(\overline{\varphi}, \overline{p}) \to (\Box\psi \to \chi)$. Since $\overline{p}$ does not occur in $\overline{\varphi}, \Box\psi \to \chi$, by admissibility of ρ, we have $\vdash_{\mathsf{SIC}} G(\overline{\varphi}) \to (\Box\psi \to \chi)$. Therefore, $\vdash_{\mathsf{SIC}} \Box\psi \to (G(\overline{\varphi}) \to \chi)$, and applying Theorem 4.4 again yields $\{\psi\} \vdash_{\mathsf{SIC}} G(\overline{\varphi}) \to \chi$. Thus, $\Gamma \vdash_{\mathsf{SIC}} G(\overline{\varphi}) \to \chi$. □

Theorem 6.12 (Admissibility Criterion).

(1) *A* Π_2*-rule* ρ *is admissible in* SIC *iff for each* $(B, \rightsquigarrow) \in \mathsf{RSub}$ *there is* $(C, \rightsquigarrow) \in \mathsf{RSub}$ *such that* $(B, \rightsquigarrow)$ *is isomorphic to a substructure of* $(C, \rightsquigarrow)$ *and* $(C, \rightsquigarrow) \models \Pi(\rho)$*.*
(2) *A* Π_2*-rule* ρ *is admissible in* $\mathsf{S^2IC}$ *iff for each* $(B, \rightsquigarrow) \in \mathsf{Con}$ *there is* $(C, \rightsquigarrow) \in \mathsf{Con}$ *such that* $(B, \rightsquigarrow)$ *is isomorphic to a substructure of* $(C, \rightsquigarrow)$ *and* $(C, \rightsquigarrow) \models \Pi(\rho)$*.*

Proof. (1) ($\Rightarrow$) Suppose ρ is admissible in SIC. It is sufficient to show that there exists a model $(C, \rightsquigarrow)$ of the theory

$$T = \mathrm{Th}(\mathsf{RSub}) \cup \{\Pi(\rho)\} \cup \Delta(B, \rightsquigarrow)$$

where $\Delta(B, \rightsquigarrow)$ is the diagram of $(B, \rightsquigarrow)$ [11, p. 68]. Suppose for a contradiction that T has no models, hence is inconsistent. Then, by compactness, there exists a quantifier-free first-order formula $\Psi(\overline{x})$ and a tuple $\overline{p_a}$ of propositional letters corresponding to $\overline{a} \in B$ such that

$$\mathrm{Th}(\mathsf{RSub}) \cup \{\Pi(\rho)\} \models \neg\Psi(\overline{p_a}) \text{ and } (B, \rightsquigarrow) \models \Psi(\overline{a}).$$

We enrich the language of $\mathsf{SIC}+\rho$ with $\{\overline{p_a}\}$. By Theorem 6.6, $\mathsf{SIC}+\rho$ is complete with respect to the algebras in RSub satisfying $\Pi(\rho)$. Therefore, by Lemma 6.8(1), $\vdash_{\mathsf{SIC}+\rho} (\neg\Psi(\overline{p_a}))^*$ where $(-)^*$ is the translation given in Definition 6.7. By admissibility, $\vdash_{\mathsf{SIC}} (\neg\Psi(\overline{p_a}))^*$. Thus, for each valuation v into B that maps p_a to a, we have $v((\neg\Psi(\overline{p_a}))^*) = 1$, so $v((\Psi(\overline{p_a}))^*) = 0$. This contradicts the fact that $(B, \rightsquigarrow) \models \Psi(\overline{a})$. Consequently, T must be consistent, and hence it has a model.

($\Leftarrow$) To prove that ρ is admissible in SIC it is sufficient to show that if there is a proof of $F(\overline{\varphi}, \overline{p}) \to \chi$, then there is a proof of $G(\overline{\varphi}) \to \chi$. Suppose $\vdash_{\mathsf{SIC}} F(\overline{\varphi}, \overline{p}) \to \chi$ with $\overline{p}$ not occurring in $\overline{\varphi}, \chi$. Assume $(B, \rightsquigarrow) \in \mathsf{RSub}$ and let v be a valuation on $(B, \rightsquigarrow)$. By assumption, there is $(C, \rightsquigarrow) \in \mathsf{RSub}$ such that $(B, \rightsquigarrow)$ is isomorphic to a substructure of $(C, \rightsquigarrow)$ and $(C, \rightsquigarrow) \models \Pi(\rho)$. Let $i : B \to C$ be the embedding. Then $v' := i \circ v$ is a valuation on $(C, \rightsquigarrow)$. For any $\overline{c} \in C$, let v'' be the valuation $(v')^{\overline{c}}_{\overline{p}}$. Since $\vdash_{\mathsf{SIC}} F(\overline{\varphi}, \overline{p}) \to \chi$, by Theorem 4.11, $v''(F(\overline{\varphi}, \overline{p}) \to \chi) = 1_C$. This means that for all $\overline{c} \in C$, we have $F(v'(\overline{\varphi}), \overline{c}) \leq v'(\chi)$. Therefore, $(C, \rightsquigarrow) \models \forall \overline{y} \Big(F(v'(\overline{\varphi}), \overline{y}) \leq v'(\chi) \Big)$. Since $(C, \rightsquigarrow) \models \Pi(\rho)$, we have $(C, \rightsquigarrow) \models G(v'(\overline{\varphi})) \leq v'(\chi)$. Thus, as $G(v'(\overline{\varphi})) \leq v'(\chi)$ in C, we have $G(v(\overline{\varphi})) \leq v(\chi)$ in B. Consequently, $v(G(\overline{\varphi}) \to \chi) = 1_B$. Applying Theorem 4.11 again yields that $\vdash_{\mathsf{SIC}} G(\overline{\varphi}) \to \chi$, and hence ρ is admissible in SIC.

(2) The proof is similar to that of (1) and uses the fact that $\mathsf{S^2IC}$ is strongly sound and complete with respect to Con. □

Corollary 6.13.

(1) (ρ6) *is admissible in* SIC *and* $\mathsf{S^2IC}$.
(2) (ρ7) *is admissible in* SIC *and* $\mathsf{S^2IC}$.

Proof. (1) For admissibility of (ρ6) in SIC apply Theorem 6.12(1) and Lemmas 6.1(1) and 5.4(1). For admissibility of (ρ6) in $\mathsf{S^2IC}$, apply Theorem 6.12(2) and Lemmas 6.1(1) and 5.4(2).

(2) For admissibility of (ρ7) in SIC apply Theorem 6.12(1) and Lemmas 6.1(2) and 5.6. For admissibility of (ρ7) in $\mathsf{S^2IC}$ apply Theorem 6.12(2) and Lemmas 6.1(2) and 5.6. □

6.3. Calculi for zero-dimensional and connected compact Hausdorff spaces

In the remainder of this section, we will consider zero-dimensional and connected compact Hausdorff models, and identify their logics. Starting with zero-dimensionality, we consider the following property, studied in [4]:

(S9) $a \prec b$ implies $\exists c : c \prec c$ and $a \prec c \prec b$.

The corresponding $\forall\exists$-statement is

(Π9) $\forall x, y, z\Big(x \rightsquigarrow y \nleq z \to \exists u : (u \rightsquigarrow u) \wedge (x \rightsquigarrow u) \wedge (u \rightsquigarrow y) \nleq z\Big)$.

Lemma 6.14. *Let* $(B, \rightsquigarrow) \in \mathsf{Com}$. *Then* $(B, \rightsquigarrow) \models$ (S9) *iff* $(B, \rightsquigarrow) \models$ (Π9).

Proof. ($\Rightarrow$) Suppose $a \rightsquigarrow b \nleq d$. Then $d \neq 1$ and $a \rightsquigarrow b \neq 0$, so $a \rightsquigarrow b = 1$. Therefore, $a \prec b$, and so by (S9), there is c such that $c \prec c$ and $a \prec c \prec b$. Thus, $(c \rightsquigarrow c) \wedge (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) = 1 \nleq d$. Consequently, $(B, \rightsquigarrow) \models$ (Π9).

($\Leftarrow$) Suppose $a \prec b$. Then $a \rightsquigarrow b = 1 \nleq 0$. Therefore, by (Π9), there is c such that $(c \rightsquigarrow c) \wedge (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) \nleq 0$, which implies $(c \rightsquigarrow c) \wedge (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) = 1$. Thus, $c \prec c$ and $a \prec c \prec b$. Consequently, $(B, \rightsquigarrow) \models$ (S9). □

The Π_2-rule corresponding to (Π9) is

$$(\rho 9) \quad \frac{(p \rightsquigarrow p) \wedge (\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$$

Theorem 6.15. (ρ9) *is admissible in* SIC *and* $\mathsf{S^2IC}$.

Proof. It is easy to see that the $(C, \prec)$ constructed in the proof of Lemma 5.4 satisfies (S9). Therefore, for admissibility in SIC, we can apply Theorem 6.12(1), Lemma 6.14, and Lemma 5.4(1); and for admissibility in $\mathsf{S^2IC}$, we can apply Theorem 6.12(2), Lemma 6.14, and Lemma 5.4(2). □

As a consequence of Theorem 6.15 we obtain:

Corollary 6.16. $\mathsf{S^2IC}$ *is strongly sound and complete with respect to the class of compingent algebras satisfying* (S9).

Following [4, Def. 4.5], we call a de Vries algebra *zero-dimensional* if it satisfies (S9), and denote the class of zero-dimensional de Vries algebras by zDeV. Let $(B, \rightsquigarrow) \in \mathsf{Com}$ satisfy (S9), and let X be the de Vries dual of $(B, \rightsquigarrow)$. It follows from de Vries duality and [4, Lem. 4.11] that X is zero-dimensional, and hence $(\mathcal{RO}(X), \rightsquigarrow)$ is a zero-dimensional de Vries algebra. Thus, each $(B, \rightsquigarrow) \in \mathsf{Com}$ satisfying (S9) embeds in a zero-dimensional de Vries algebra. We recall from Section 2 that zero-dimensional compact Hausdorff spaces are called Stone spaces, and denote the class of Stone spaces by Stone. As a consequence of Corollary 6.16 we then have:

Theorem 6.17.

(1) $\mathsf{S^2IC}$ *is strongly sound and complete with respect to* zDeV.
(2) $\mathsf{S^2IC}$ *is strongly sound and complete with respect to* Stone.

Turning to connectedness, consider the following property:

(S10) $a \prec a$ implies $a = 0$ or $a = 1$.

Clearly $(B, \rightsquigarrow) \in \mathsf{Com}$ satisfies (S10) iff $a \rightsquigarrow a \le \Box a \vee \Box\neg a$ holds in $(B, \rightsquigarrow)$. Therefore, $(B, \rightsquigarrow)$ satisfies (S10) iff $(B, \rightsquigarrow) \models$ (C), where (C) is the formula

(C) $(\varphi \rightsquigarrow \varphi) \to (\Box\varphi \vee \Box\neg\varphi)$.

Definition 6.18. The *connected symmetric strict implication calculus* $\mathsf{CS^2IC}$ is the extension of $\mathsf{S^2IC}$ with the axiom (C).

We call $(A, \rightsquigarrow) \in \mathsf{S^2IA}$ *connected* if $(A, \rightsquigarrow)$ satisfies $a \rightsquigarrow a \le \Box a \vee \Box\neg a$ for each $a \in A$. Let $\mathsf{CS^2IA}$ be the subvariety of $\mathsf{S^2IA}$ consisting of connected symmetric strict implication algebras. We also call a compingent algebra *connected* if it satisfies (S10), and denote the class of connected compingent algebras by CCom. As a simple consequence of Theorems 5.2 and 6.6 we have:

Corollary 6.19.

(1) $\mathsf{CS^2IC}$ *is strongly sound and complete with respect to* $\mathsf{CS^2IA}$.
(2) $\mathsf{CS^2IC}$ *is strongly sound and complete with respect to* CCom.

Lemma 6.20. *A compingent algebra* $(B, \prec)$ *satisfies* (S10) *iff its dual compact Hausdorff space* X *is connected.*

Proof. If X is connected, then $\varnothing, X$ are the only clopen subsets of X. Therefore, for $U \in \mathcal{RO}(X)$, we have $U \prec U$ implies $U = \varnothing$ or $U = X$. Thus, $(\mathcal{RO}(X), \prec)$ satisfies (S10). Since $(B, \prec)$ is isomorphic to a subalgebra of $(\mathcal{RO}(X), \prec)$, we conclude that $(B, \prec)$ satisfies (S10).

Conversely, let U be clopen in X. Then $U = \bigcup\{\beta(a) : \beta(a) \subseteq U\}$ and the family $\{\beta(a) : \beta(a) \subseteq U\}$ is up-directed. Because U is compact, there is $a \in B$ such that $\beta(a) = U$. As $\beta : B \to \mathcal{RO}(X)$ is an embedding and $(B, \prec)$ satisfies (S10), $U = \beta(0) = \varnothing$ or $U = \beta(1) = X$. Thus, X is connected. □

As an immediate consequence we obtain:

Lemma 6.21. *A de Vries algebra* $(B, \prec)$ *satisfies* (S10) *iff its de Vries dual* X *is connected.*

We call a de Vries algebra *connected* if it satisfies (S10), and denote the class of connected de Vries algebras by cDeV. By Lemma 6.21, each $(B, \prec) \in \mathsf{CCom}$ embeds in a connected de Vries algebra. Let cKHaus be the class of connected compact Hausdorff spaces. Then Corollary 6.19 and Lemma 6.21 imply:

Theorem 6.22.

(1) $\mathsf{CS^2IC}$ *is strongly sound and complete with respect to* cDeV.

(2) $\mathsf{CS^2IC}$ *is strongly sound and complete with respect to* cKHaus.

The table below summarizes our completeness results.

Logic	Complete with respect to
SIC	SIA; RSub
$\mathsf{S^2IC}$	$\mathsf{S^2IA}$; Con; Com; DeV; zDeV
	KHaus; Stone
$\mathsf{CS^2IC}$	$\mathsf{CS^2IA}$; CCom; cDeV; cKHaus

Remark 6.23. By Theorems 5.2 and 5.10, $\mathsf{S^2IC}$ is strongly sound and complete with respect to Con and DeV. Thus, the logic of contact algebras is the same as the logic of de Vries algebras. On the other hand, the Π_2-theories (the sets of valid Π_2-rules) of Con and DeV are obviously different as the Π_2-rules ($\rho 6$) and ($\rho 7$) belong to the latter, and hence to the theory of compact Hausdorff spaces, but not to the former. This generates an interesting methodological question of what the right logical formalism should be to reason about compact Hausdorff spaces. Should we be concerned only with the logics or should we also consider the theories of Π_2-rules? Although in this paper we are only concerned with logics, our results suggest that a theory of Π_2-rules may be a more appropriate framework to reason about compact Hausdorff spaces. We leave it as a future work to develop the Π_2-theory for compact Hausdorff spaces together with the general theory of such calculi.

7. Comparison with relevant work

In this final section we compare our approach to that of Balbiani et al. [1]. Namely, we show how to translate fully and faithfully the language $\mathsf{L}(\mathsf{C}, \leq)$ of [1] into our language. We start by recalling the concept of contact relation, one of the key concepts of region-based theory of space; see, e.g., [27]. A binary relation C on a Boolean algebra B is a *precontact relation* if it satisfies:

(C1) $a\mathsf{C}b \Rightarrow a, b \neq 0$.
(C2) $a\mathsf{C}(b \vee c) \Leftrightarrow a\mathsf{C}b$ or $a\mathsf{C}c$.
(C3) $(a \vee b)\mathsf{C}c \Leftrightarrow a\mathsf{C}c$ or $b\mathsf{C}c$.

A precontact relation is a *contact relation* if it satisfies:

(C4) $a \neq 0$ implies $a\mathsf{C}a$.
(C5) $a\mathsf{C}b$ implies $b\mathsf{C}a$.

As was pointed out in [6, Rem. 2.5] (cf. Remark 3.2), there is a one-to-one correspondence between subordinations and precontact relations. Namely, if $\prec$ is a subordination, then the relation $\mathsf{C}_\prec$ defined by $a\mathsf{C}_\prec b$ iff $a \not\prec \neg b$ is a precontact relation; if C is a precontact relation, then the relation $\prec_\mathsf{C}$ defined by $a \prec_\mathsf{C} b$ iff $a \not\mathsf{C} \neg b$ is a subordination; and this correspondence is one-to-one. Moreover, a subordination $\prec$ satisfies (S5) iff the corresponding precontact relation $\mathsf{C}_\prec$ satisfies (C4), and $\prec$ satisfies (S6) iff $\mathsf{C}_\prec$ satisfies (C5). Thus, contact relations are in one-to-one correspondence with subordinations satisfying (S5) and (S6).

On regular open sets of a compact Hausdorff space X the contact relation is defined by $U\mathsf{C}V$ iff $\mathsf{Cl}(U) \cap \mathsf{Cl}(V) \neq \varnothing$. If R is a reflexive and symmetric relation on a set X, then the contact relation C_R is defined on $\mathcal{P}(X)$ by $U\mathsf{C}_R V$ iff $R[U] \cap V \neq \varnothing$.

We next recall that the formulas of the language $\mathsf{L}(\mathsf{C}, \leq)$ are built from atomic formulas using Boolean connectives $\neg, \wedge, \vee, \rightarrow, \bot, \top$; atomic formulas are of the form $t\mathsf{C}s$ and $t \leq s$, where t, s are Boolean terms

(C stands for the contact relation and $\leq$ for the inclusion relation). In turn, Boolean terms are built from Boolean variables using Boolean operations $\sqcap, \sqcup, (-)^*, 0, 1$.

As usual, a *Kripke frame* is a pair (W, R), where W is a nonempty set and R is a binary relation on W, and a *valuation* is a map v from the set of Boolean variables to the powerset $\mathcal{P}(W)$. It extends to the set of all Boolean terms as follows:

$$\begin{aligned} v(t \sqcap s) &= v(t) \cap v(s), \\ v(t \sqcup s) &= v(t) \cup v(s), \\ v(t^*) &= W \setminus v(t), \\ v(0) &= \varnothing, \\ v(1) &= W. \end{aligned}$$

A *Kripke model* is a triple (W, R, v) consisting of a Kripke frame (W, R) and a valuation v. Atomic formulas are interpreted in (W, R, v) as follows:

$$\begin{aligned} (W, R, v) \models (t \leq s) &\iff v(t) \subseteq v(s), \\ (W, R, v) \models (t \mathsf{C} s) &\iff R[v(t)] \cap v(s) \neq \varnothing. \end{aligned}$$

Complex formulas are then interpreted by the induction clauses for propositional connectives.

In [1, Sec. 6] the authors define the propositional calculus PWRCC in the language $\mathsf{L}(\mathsf{C}, \leq)$ and prove that PWRCC is sound and complete with respect to the class of Kripke frames where the binary relation R is reflexive and symmetric. Such Kripke frames are closely related to contact algebras. Namely, as we already pointed out in Sections 2 and 5, the following lemma holds.

Lemma 7.1.

(1) *Suppose (W, R) is a reflexive and symmetric Kripke frame. Define $\prec_R$ on $\mathcal{P}(W)$ by $U \prec_R V$ iff $R[U] \subseteq V$. Then $(\mathcal{P}(W), \prec_R)$ is a contact algebra.*

(2) *Suppose $(B, \prec)$ is a contact algebra and (X, R) is the dual of $(B, \prec)$. Then (X, R) is a reflexive and symmetric Kripke frame, and the Stone map $\beta : B \to \mathcal{P}(X)$, given by $\beta(a) = \{x \in X \mid a \in x\}$, is an embedding of $(B, \prec)$ into $(\mathcal{P}(X), \prec_R)$.*

We next translate $\mathsf{L}(\mathsf{C}, \leq)$ into our language $\mathcal{L}$. We identify the set of Boolean variables of $\mathsf{L}(\mathsf{C}, \leq)$ with the set of propositional letters of $\mathcal{L}$. Then Boolean terms can be translated into formulas of $\mathcal{L}$ as follows:

$$\begin{aligned} a^T &= a, \text{ for a Boolean variable } a, \\ (t \sqcap s)^T &= t^T \wedge s^T, \\ (t \sqcup s)^T &= t^T \vee s^T, \\ (t^*)^T &= \neg(t^T), \\ 0^T &= \bot, \\ 1^T &= \top. \end{aligned}$$

For atomic formulas, we define:

$$\begin{aligned}(t \le s)^T &= \Box(t^T \to s^T),\\ (t\mathsf{C}s)^T &= \neg(t^T \rightsquigarrow \neg s^T).\end{aligned}$$

Finally, complex formulas are translated inductively as follows:

$$\begin{aligned}(\neg\varphi)^T &= \neg\varphi^T,\\ (\varphi \wedge \psi)^T &= \varphi^T \wedge \psi^T,\\ (\varphi \vee \psi)^T &= \varphi^T \vee \psi^T,\\ (\varphi \to \psi)^T &= \varphi^T \to \psi^T,\\ \bot^T &= \bot,\\ \top^T &= \top.\end{aligned}$$

Theorem 7.2. *For any formula* φ *of* $\mathsf{L}(\mathsf{C}, \le)$*, we have*

$$\mathsf{PWRCC} \vdash \varphi \textit{ iff } \mathsf{S^2IC} \vdash \varphi^T.$$

Proof. By [1, Cor. 6.1], PWRCC is sound and complete with respect to the class of reflexive and symmetric Kripke frames (W, R); and by Theorem 5.2, $\mathsf{S^2IC}$ is sound and complete with respect to the class of contact algebras. Given a Kripke model (W, R, v), the valuation v of Boolean variables of $\mathsf{L}(\mathsf{C}, \le)$ into $\mathcal{P}(W)$ can be seen as a valuation of propositional letters of $\mathcal{L}$ into the algebra $(\mathcal{P}(W), \rightsquigarrow_R)$.

Claim. $(W, R, v) \models \varphi$ iff $(\mathcal{P}(W), \rightsquigarrow_R, v) \models \varphi^T$.

Proof of Claim. For a Boolean term t, we have $v(t) = v(t^T) \subseteq W$. If φ is an atomic formula of the form $t \le s$, then

$$\begin{aligned}(W, R, v) \models \varphi &\text{ iff } v(t) \subseteq v(s)\\ &\text{ iff } v(t^T) \le v(s^T) \text{ in } \mathcal{P}(W)\\ &\text{ iff } (\mathcal{P}(W), \rightsquigarrow_R, v) \models t^T \to s^T\\ &\text{ iff } (\mathcal{P}(W), \rightsquigarrow_R, v) \models \Box(t^T \to s^T)\\ &\text{ iff } (\mathcal{P}(W), \rightsquigarrow_R, v) \models \varphi^T.\end{aligned}$$

If φ is an atomic formula of the form $t\mathsf{C}s$, then

$$\begin{aligned}(W, R, v) \models \varphi &\text{ iff } R[v(t)] \cap v(s) \ne \varnothing\\ &\text{ iff } R[v(t)] \nsubseteq W \setminus v(s)\\ &\text{ iff } R[v(t^T)] \nsubseteq v(\neg s^T)\\ &\text{ iff } (\mathcal{P}(W), \rightsquigarrow_R, v) \models \neg(t^T \rightsquigarrow \neg s^T)\\ &\text{ iff } (\mathcal{P}(W), \rightsquigarrow_R, v) \models \varphi^T.\end{aligned}$$

Finally, if φ is a complex formula, then a straightforward induction completes the proof. □

Now, if $\mathsf{PWRCC} \nvdash \varphi$, then there is a reflexive and symmetric Kripke model (W, R, v) refuting φ. By the Claim, φ^T is refuted in $(\mathcal{P}(W), \rightsquigarrow_R, v)$. Therefore, $\mathsf{S^2IC} \nvdash \varphi^T$. Conversely, if $\mathsf{S^2IC} \nvdash \varphi^T$, then there is a contact algebra $(B, \prec)$ and a valuation v on $(B, \prec)$ refuting φ^T. By Lemma 7.1(2), φ^T is refuted in $(\mathcal{P}(X), \rightsquigarrow_R, v)$. By the Claim, φ is refuted in (X, R, v). Thus, $\mathsf{PWRCC} \nvdash \varphi$. $\square$

As was pointed out to us by D. Vakarelov, our language, like the language of [1], admits a translation tr into the basic language of modal logic enriched with the universal modality. We conclude the paper by spelling out this connection.

We recall that K denotes the basic modal logic, $\mathsf{KT} := \mathsf{K} + (\Box p \to p)$, $\mathsf{KTB} := \mathsf{KT} + (p \to \Box\Diamond p)$, and $\mathsf{S5} := \mathsf{KTB} + (\Box p \to \Box\Box p)$ (see, e.g., [7, Sec. 4.1]). We also recall that the universal modality $[\forall]$ is an $\mathsf{S5}$-modality satisfying $[\forall]p \to \Box p$. Let $\mathsf{KT_U}$, $\mathsf{KTB_U}$, and $\mathsf{S5_U}$ be the extensions of KT, KTB, and $\mathsf{S5}$ with the universal modality (see, e.g., [7, Sec. 7.1]).

It is well known that all the above logics are Kripke complete and have the finite model property (see, e.g., [7]). In particular, it is known that $\mathsf{KT_U}$ has the finite model property with respect to the finite reflexive Kripke frames enriched with the universal relation, that $\mathsf{KTB_U}$ has the finite model property with respect to the finite reflexive and symmetric frames enriched with the universal relation, and that $\mathsf{S5_U}$ has the finite model property with respect to the finite reflexive, symmetric, and transitive frames enriched with the universal relation (see, e.g., [20]).

Consider the following translation tr from our language into the modal language enriched with the universal modality:

$$\begin{aligned} tr(p) &= p, \\ tr(\neg\varphi) &= \neg tr(\varphi), \\ tr(\varphi \wedge \psi) &= tr(\varphi) \wedge tr(\psi), \\ tr(\varphi \rightsquigarrow \psi) &= [\forall](tr(\varphi) \to \Box tr(\psi)). \end{aligned}$$

Theorem 7.3. *Let φ be a formula of $\mathcal{L}$.*

(1) $\mathsf{SIC} \vdash \varphi$ *iff* $\mathsf{KT_U} \vdash tr(\varphi)$.
(2) $\mathsf{S^2IC} \vdash \varphi$ *iff* $\mathsf{KTB_U} \vdash tr(\varphi)$.
(3) $\mathsf{S^2IC} \vdash \varphi$ *iff* $\mathsf{S5_U} \vdash tr(\varphi)$.

Proof. (1) Let $(B, \prec) \in \mathsf{RSub}$ and let (X, R) be the dual of $(B, \prec)$. By Lemma 2.2(1), R is reflexive. Let v be a valuation on $\mathsf{Clop}(X)$. By defining the universal relation on X, we can view (X, R, v) as a model of $\mathsf{KT_U}$. We prove that for each formula ψ and $x \in X$,

$$x \in v(\psi) \text{ iff } (X, R, v), x \models tr(\psi). \tag{1}$$

This can be done by induction on the complexity of ψ, and the only nontrivial case is $\psi = \chi \rightsquigarrow \xi$. If $x \notin v(\chi \rightsquigarrow \xi)$, then $R[v(\chi)] \nsubseteq v(\xi)$. So there exist $y, z \in X$ such that $y \in v(\chi)$, yRz and $z \notin v(\xi)$. By the induction hypothesis, $(X, R, v), y \models tr(\chi)$ and $(X, R, v), z \not\models tr(\xi)$. Thus, $(X, R, v), y \not\models tr(\chi) \to \Box tr(\xi)$, and so $(X, R, v), x \not\models [\forall](tr(\chi) \to \Box tr(\xi))$. The converse implication is proved similarly. From (1) we obtain that

$$(\mathsf{Clop}(X), \rightsquigarrow_R, v) \models \psi \text{ iff } (X, R, v) \models tr(\psi). \tag{2}$$

Now, if $\mathsf{SIC} \nvdash \varphi$, then by Theorem 4.11, there is $(B, \prec) \in \mathsf{RSub}$ such that $(B, \rightsquigarrow)$ refutes φ. Therefore, there is a valuation v on $\mathsf{Clop}(X)$ such that $(\mathsf{Clop}(X), \rightsquigarrow_R, v) \not\models \varphi$. By (2), $(X, R, v) \not\models tr(\varphi)$, and hence $\mathsf{KT_U} \nvdash tr(\varphi)$. Conversely, if $\mathsf{KT_U} \nvdash tr(\varphi)$, then there is a finite reflexive model (X, R, v) with the universal relation such that $(X, R, v) \not\models tr(\varphi)$. Since (X, R) is finite, we may view it as the dual of $(\mathcal{P}(X), \prec_R)$. By (2), $(X, R, v) \not\models tr(\varphi)$ implies $(\mathcal{P}(X), \rightsquigarrow_R, v) \not\models \varphi$. Since $(\mathcal{P}(X), \prec_R) \in \mathsf{RSub}$, we conclude that $\mathsf{SIC} \nvdash \varphi$.

The proof of (2) is similar to (1) but uses Theorem 5.2.

The proof of (3) is similar to (1) and (2) but uses Theorem 5.8. □

The theorem above also indicates that unlike the classical modal case, where reflexivity, symmetry, and transitivity are all expressible, in our system we can only express reflexivity and symmetry, but not transitivity as $\mathsf{S^2IC}$ is complete with respect to (X, R) where R is an equivalence relation. Thus, our language cannot distinguish between $\mathsf{KTB_U}$ and $\mathsf{S5_U}$. This is yet another motivation to investigate non-standard rules and inductive classes of subordination algebras further.

Declaration of Competing Interest

There is no competing interest.

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