ERDŐS GRAPHS RESOLVE FINE'S CANONICITY PROBLEM

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Abstract. We show that there exist $2^{\aleph_0}$ equational classes of Boolean algebras with operators that are not generated by the complex algebras of any first-order definable class of relational structures. Using a variant of this construction, we resolve a long-standing question of Fine, by exhibiting a bimodal logic that is valid in its canonical frames, but is not sound and complete for any first-order definable class of Kripke frames (a monomodal example can then be obtained using simulation results of Thomason). The constructions use the result of Erdős that there are finite graphs with arbitrarily large chromatic number and girth.

Keywords: Boolean algebras with operators, modal logic, random graphs, canonical extension, elementary class, variety.

§1. The problem and its history. This paper describes a solution to a problem that has intrigued algebraic and modal logicians for several decades. It can be formulated as a question about systems of propositional intensional logic, or as one about equationally definable classes of Boolean algebras with additional operators. It concerns an intimate relationship between the first-order logic of relational structures and the equational logic of their Boolean algebras of subsets.

Jónsson and Tarski introduced in [36, 37] the notion of a Boolean algebra with additive operators (BAO), thereby laying the foundation for extensive studies of cylindric algebras [27, 25, 28], relation algebras [38, 34, 29], and numerous varieties of algebraic models for modal, temporal, and other kinds of intensional logic [18, 5, 40, 20, 2]. They generalized the Stone representation of Boolean algebras by showing that any BAO $\mathcal{A}$ has a "perfect" extension $\mathcal{A}^\sigma$, nowadays called the *canonical extension*

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The authors feel that morally Paul Erdős deserves to be named as a co-author of this paper, but recognize that this might be mistaken for an attempt to award themselves Erdős number 1.

of $\mathcal{A}$, which is a complete atomic BAO whose operators are completely additive. They defined a certain relational structure $\mathcal{A}_+$ associated with $\mathcal{A}^\sigma$, and proved that $\mathcal{A}^\sigma$ itself is isomorphic to an algebra based on the full powerset of $\mathcal{A}_+$. In general the powerset of any relational structure S defines a BAO S^+, called the *complex algebra*[1] of S. Each of its n-ary operators is constructed from one of the $n+1$-ary relations of S. Thus $\mathcal{A}^\sigma \cong (\mathcal{A}_+)^+$. The structure $\mathcal{A}_+$ is called the *canonical structure* of the algebra $\mathcal{A}$.

Jónsson and Tarski initiated the study of properties that are preserved in passing from an algebra $\mathcal{A}$ to its canonical extension $\mathcal{A}^\sigma$ (or $(\mathcal{A}_+)^+$), proving that they include any property expressed by an equation that does not involve Boolean complementation. Now the class of all algebras satisfying a given equation, or a set of equations, is called a *variety*, so we can express these preservation results by saying that certain equational properties define varieties that are *closed under canonical extensions*. It was demonstrated in [37] that a number of such equational properties of a *unary* operator of a complex algebra S^+ are equivalent to simple first-order properties of the corresponding *binary* relation of S, like reflexivity, transitivity, symmetry and functionality. Putting these observations together showed that a BAO $\mathcal{A}$ satisfying a certain equation is isomorphically embeddable into the complex algebra $(\mathcal{A}_+)^+$ which also satisfies the equation, and so the structure $\mathcal{A}_+$ satisfies the corresponding first-order property. This resulted in representations of particular kinds of BAO as subalgebras of complex algebras that were in turn defined by conditions on their underlying structures. Thus a *closure* algebra, which has a unary operator satisfying the Kuratowski equations for a topological-closure operator, was represented as an algebra of subsets of a *quasi-ordered* structure. Certain two-dimensional cylindric algebras (without diagonals) were similarly represented over structures comprising a pair of commuting equivalence relations.

Independently of all this, a decade or so later modal logicians began to study structures called *Kripke frames*[2]. A modal logic L is said to be *determined* by a class $\mathcal{C}$ of frames if L is both sound and complete for validity in the members of $\mathcal{C}$. This means that any given formula is an L-theorem if, and only if, it is valid in every frame belonging to $\mathcal{C}$. The Kripke semantics provided an attractive model theory that seemed more manageable than the previous algebra-based semantics and which has been of lasting influence, both mathematically and philosophically. One of the reasons for its early success was that well known logical systems

[1] At that time the word "complex" was still used in algebra to mean "subset", a terminology introduced into group theory by Frobenius in the 1880s.

[2] See [22] for the historical background to the origin of this concept in the work of Kripke, Kanger, Hintikka, Montague, and others.

were shown to be characterized by natural first-order properties of their frames. Thus Lewis's system S4 is determined by the class of quasi-orderings, and S5 by the class of equivalence relations. Different classes of frames can determine the same logic: for example, S4 is also determined by the class of partial orderings, the class of reflexive-transitive closures of tree orderings, and the class of finite quasi-orderings (but not the finite partial orderings). We will say that a logic L is *elementarily determined* if there is at least one class determining L that is elementary, i.e., is axiomatised by some first-order sentences. It is quite possible for L to be determined by some elementary class while at the same time the class of *all* frames validating L is not elementary.

Proofs of elementary determination for a number of logics, including S4, S5 and the system T, could have been obtained by adapting the Jónsson–Tarski methodology, but this was not noticed at the time.[3] Instead a technique was developed that uses *canonical frames*, introduced by Lemmon and Scott [42], and independently by Cresswell [6] and Makinson [44]. These are structures whose points are maximally consistent sets of formulas, and their use is an extension of the method of completeness proof due to Henkin [26]. A canonical frame for a logic L carries a special interpretation that falsifies all non-theorems of L. Sometimes it carries other interpretations that falsify L-theorems as well. But in more tractable cases, the proof theory of L can be used to show L is a *canonical logic*, meaning that it is valid in all its canonical frames. A canonical logic is determined by these canonical frames alone.

Now the notion of validity in a frame is intrinsically second-order in nature. Indeed, Thomason [47] gave a semantic reduction of monadic second-order logic to propositional modal logic. Also work of Blok [3] showed that there are continuum many modal logics that are not determined by any class of frames at all, let alone an elementary one. Nonetheless, many logics of mathematical and philosophical interest were shown to be frame-determined by showing that their canonical frames satisfy some first-order conditions that enforce validity of the theorems of the logic. This gave many proofs of canonicity which at the same time showed that the logic concerned was elementarily determined. Moreover, the only examples of non-canonical logics that came to light were ones whose axioms expressed non-elementary properties of structures, such as well-foundedness or discreteness of orderings. An explanation for this was soon found in the following seminal theorem of Fine [10]:

(1) if a modal logic is determined by some elementary class of frames, then it is validated by its canonical frames.

[3]Why did Tarski not develop the Kripke semantics himself, given his work on BAOs and his earlier work with McKinsey [45] on closure-algebraic models of S4? For discussion of this question see [22].

Fine asked whether the converse was true. If a logic is canonical, must it be elementarily determined? An affirmative answer would completely account for the observed propinquity between these two conceptually quite different notions.

Fine's theorem was extended by the first-named author in two directions. First, the conclusion was strengthened to show that if a logic is determined by some elementary conditions, then it is always determined by elementary conditions that are satisfied by its canonical frames (see [16]). Secondly, the result was formulated algebraically and established for varieties of BAOs of any kind. The link between the worlds of logic and algebra here is that if S is a canonical frame for a logic L, then S is isomorphic to the canonical structure $\mathcal{A}_+$ of some *free* algebra $\mathcal{A}$ in the variety of all BAOs that validate equations corresponding to the theorems of L, and so $S^+ \cong \mathcal{A}^\sigma$. Now if $\mathcal{C}$ is a class of relational structures of the same type, let $\mathrm{Var}\,\mathcal{C}$ be the smallest variety that includes the class $\mathcal{C}^+ = \{S^+ : S \in \mathcal{C}\}$ of complex algebras of members of $\mathcal{C}$. $\mathrm{Var}\,\mathcal{C}$ is just the class of all models of the equational theory of $\mathcal{C}^+$, or equivalently, the closure of $\mathcal{C}^+$ under homomorphic images, subalgebras, and direct products. It will be called the variety *generated by* $\mathcal{C}$. Then the algebraic analogue of Fine's theorem is the following result [17]:

(2) if a variety $\mathcal{V}$ is generated by some elementary class of structures, then it is closed under canonical extensions, i.e., $\mathcal{A} \in \mathcal{V}$ implies $\mathcal{A}^\sigma \in \mathcal{V}$.

A variety of BAOs will be called *canonical* if it is closed under canonical extensions, and *elementarily generated* if it is generated by some elementary class of structures. It turns out that for $\mathcal{V}$ to be elementarily generated, it is enough that $\mathcal{V} = \mathrm{Var}\,\mathcal{C}$ for some class $\mathcal{C}$ that is closed under ultraproducts. The strengthened version of Fine's theorem becomes the result [19] that

(3) if a variety $\mathcal{V}$ is generated by some ultraproducts-closed class of structures, then it is generated by an elementary class that includes the class $\{\mathcal{A}_+ : A \in \mathcal{V}\}$ of all canonical structures of members of $\mathcal{V}$.

The algebraic version of Fine's question is the converse of (2): is every canonical variety elementarily generated? Over the years there have been many partial confirmations of these converse questions:

- A modal analysis by Sahlqvist [46], generalized to arbitrary types of BAOs by de Rijke and the third author [7] and analysed further by Jónsson [35], gives a syntactic scheme producing infinitely many equations/formulas, each of which defines a canonical variety, and whose frame-validity is equivalent to an explicit first-order condition.

- Jónsson [35] showed that an equation of the form $t(x+y) = t(x)+t(y)$ defines a canonical variety whenever t is a unary term whose interpretation commutes with canonical extensions. This implies that a modal axiom of the form $\varphi(p \vee q) \leftrightarrow \varphi(p) \vee \varphi(q)$ is canonical whenever $\varphi(p)$ is a positive formula. The third author [49] showed that logics with such axioms are elementarily determined.
- Fine [11] proved the converse of (1) for any modal logic that is determined by a class of transitive frames that is closed under subframes. Zakharyaschev [52] extended this to logics determined by a class of transitive frames that is closed under *cofinal* subframes. Wolter [51] removed the transitivity restriction in Fine's result, showing that the converse of (1) holds for logics determined by any class that is closed under subframes.
- Wolter [50] proved the converse of (1) for all normal extensions of linear tense logic.
- In the theory of cylindric and relation algebras, there are a number of infinite families of varieties that have been shown to be canonical by various structural means. They include the varieties $\mathbf{S}\mathfrak{Nr}_\beta\mathbf{CA}_\alpha$ of neat β-dimensional subreducts of α-dimensional cylindric algebras, defined by Henkin; the varieties $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_\alpha$ of subalgebras of relation algebra reducts of α-dimensional cylindric algebras, due to Henkin and Tarski; and the varieties $\mathbf{RA}_n$ of subalgebras of atomic non-associative algebras with n-dimensional bases, due to Maddux. All of these have subsequently been confirmed to be elementarily generated [23].
- A modal formula is called *r-persistent* if it is validated by a Kripke frame $S = (W, R)$ whenever it is validated by some subalgebra of S^+ that is a base for a Hausdorff topology on S in which sets of the form $\{y : xRy\}$ are closed. Every logic with r-persistent axioms is canonical and hence is determined by its validating frames. Lachlan [41] showed that the class of validating frames for an r-persistent formula is definable by a first-order sentence.
- The converse of (2) holds for any variety that contains a complex algebra S^+ whenever it contains the subalgebra generated by the atoms (singletons) of S^+ [21]. This also implies the just-mentioned result that r-persistent logics are elementarily determined.
- Gehrke, Harding and the third author have recently shown that any variety that is closed under MacNeille completions is both canonical and elementarily generated [13].

Despite all that positive evidence, this paper shows that the converses of (1) and (2) are not true in general. Continuum many canonical varieties are defined, none of which are generated by any elementary class. They consist of BAOs with two unary operators, one of which models the modal logic S5. All of the varieties are generated by their finite members, so the corresponding logics have the finite model property. The universal theory of each variety is the same as the universal theory of its finite members. One variety is shown to have a decidable universal theory.

Here is the essential idea behind these examples. It is known that a variety $\mathcal{V}$ must fail to be elementarily generated if there exists a sequence of *finite* structures whose complex algebras are all in $\mathcal{V}$, and an ultraproduct of the sequence whose complex algebra is not in $\mathcal{V}$ (this can be shown using (3); see proposition 2.17 and the proof of theorem 2.18 below). It has long been known that there are varieties for which there is such a sequence, but until now no such variety was known to be canonical. The construction given here involves an application of a famous piece of graph theory. Erdős showed in [9] that there are finite graphs with arbitrarily large chromatic number and girth, the girth being the length of the shortest cycle in the graph. This may seem counter-intuitive, in that a lack of short cycles should make it easier to colour a graph with few colours. Nonetheless, by a revolutionary new probabilistic technique, Erdős was able to show that there is a sequence of finite graphs whose n-th member G_n cannot be coloured with n or fewer colours and has no cycle of length n or less. But an ultraproduct of such a sequence will have no cycles at all, which implies that it can be coloured using only two colours! The task then is to find a set of conditions that are incompatible with 2-colouring, are satisfied by the algebras G_n^+, and which generate a canonical variety.

The solution has a connection to a canonical modal logic, studied by Hughes [33], whose validating frames are precisely those directed graphs in which the children of any node have no finite colouring. This is not a first-order condition, but the logic is also *elementarily* determined by the class of graphs whose edge relation R satisfies $\forall x \exists y(xRyRy)$, meaning that every node has a reflexive child. The modal axioms for Hughes's logic impose this elementary condition on canonical frames, and the existence of reflexive points (yRy) ensures validity of the axioms. Note that a graph with a reflexive point cannot be coloured at all.

Here we also use conditions that impose reflexive points on canonical structures $\mathcal{A}_+$, but there is a fundamental difference. A canonical structure is now essentially the disjoint union of a family of directed graphs, and it is only the *infinite* members of the family that are required to have a reflexive point to ensure canonicity. This is a non-elementary requirement.

Ultraproducts of Erdős graphs were introduced into algebraic logic by Hirsch and the second author (see [30] and [29, chapter 14]), who used them to give a negative answer to Maddux's question from [43] of whether the collection $\{S : S^+ \in \mathbf{RRA}\}$ of structures whose complex algebra belongs to the variety of representable relation algebras constitutes an elementary class. Random graph theory has also been used by the last two authors in [32] to show that there are canonical varieties (including **RRA**) that cannot be axiomatised by equations that are *individually* preserved by canonical extensions. The results of [32] show that the varieties presented in the current paper also have this property.

The language of this paper is for the most part algebraic, and we take advantage of ideas from duality theory and the theory of discriminator varieties to present a streamlined treatment. But, as the historical aspects of this introduction indicate, the work is addressed to two communities of interest, the logical and the algebraic, each with its own language, range of problems, and style of thinking. In recognition of this, we have included a brief account of the modal approach, exhibiting a bimodal logic EG that is validated by its canonical frames but not sound and complete for any elementary class of frames. We can then obtain a unimodal example using simulation techniques of Thomason.

We believe that the algebraic form of the argument will be about as straightforward to algebraists as the modal form will be to modal logicians. We have chosen to present the algebraic side of the story first, because a fuller account of the modal versions of our results, including further examples of canonical logics that are not elementarily determined, will be presented in a companion paper [24]. But the two versions are more or less independent and can be read in either order.

Graphs. In this paper, a *graph* will be a pair $G = (V, E)$, where V is a non-empty set of 'vertices' and E is an irreflexive symmetric binary 'edge' relation on V. A set $S \subseteq V$ is said to be *independent* if for all $x, y \in S$ we have $(x, y) \notin E$. For an integer k, a *k-colouring* of G is a partition of V into k independent sets. The *chromatic number* of G is the smallest k for which it has a k-colouring, and ∞ if it has no k-colouring for any finite k. A *cycle* in G is a sequence $(x_1, \ldots, x_k)$ of distinct nodes of V, such that $k \geq 3$ and (x_1, x_2), ..., (x_{k-1}, x_k), (x_k, x_1) are all in E.[4] The *length* of the cycle is k. By definition, no graph has cycles of length < 3. It is well known that a graph has chromatic number at most two if, and only if, it has no cycles of odd length. For a proof, see, e.g., [8, proposition 1.6.1]. The result holds for both finite and infinite graphs; the implicit assumption in [8, p. 2] that graphs are finite is not needed in the proof in [8].

We often identify (notationally) a graph, algebra, structure, or frame, with its domain — for example, in the above context, we will often write G for V.

[4]In graph theory, $(x_1, \ldots, x_k)$, $(x_2, \ldots, x_k, x_1)$, and $(x_k, \ldots, x_1)$ are regarded as the same cycle; but this is not important here.

§2. The algebraic approach. We now give a detailed presentation of the algebraic approach. We begin with a rundown of the necessary concepts and notation, and then we review a general method by which we may prove a variety to be canonical. Much or all of this material will be familiar to algebraists. It will then be quite easy to show that the variety we introduce in §2.3 is canonical but not elementarily generated.

2.1. Boolean algebras with operators (BAOs). We assume familiarity with basic ideas from model theory and universal algebra, such as the notions of homomorphism, product, subalgebra, ultraproduct, ultrapower, ultraroot, equation, universal formula, and equational class (variety). We also presuppose some acquaintance with Boolean algebra theory, including notions such as atom, atomicity, completeness, ideal and (ultra)filter, and Stone representation theory. Readers may consult, e.g., [2, chapter 5], [4], [31], or [29, chapter 2] for background.

A similarity type L of BAOs will consist of the boolean function symbols $+, -$ and the constants $0, 1$, plus additional function symbols for operators. An L-BAO is an L-structure $\mathcal{A}$ whose reduct to the signature $\{+, -, 0, 1\}$ is a boolean algebra and in which the interpretations of the additional function symbols are 'operators': i.e., normal (taking value zero whenever any argument is zero), and additive (hence monotonic) in each argument. We write $x \cdot y$ for $-(-x + -y)$. We often use the same notation for a symbol in L and its interpretation in an L-BAO. Given L-BAOs $\mathcal{A}_i$ $(i \in I)$, and an ultrafilter D on I, we write $\prod_D \mathcal{A}_i$ for the ultraproduct of the $\mathcal{A}_i$ over D. $\mathbf{S}, \mathbf{P}, \mathbf{Pu}, \mathbf{Ru}$ denote closure of a class under isomorphic copies of: subalgebras, products, ultraproducts, and ultraroots, respectively.

A *discriminator* on an L-BAO $\mathcal{A}$ is a unary function d on $\mathcal{A}$ such that $d(0) = 0$ and $d(x) = 1$ for all non-zero x in $\mathcal{A}$. A class $\mathcal{K}$ of L-BAOs is a *discriminator class* if some L-term $t(x)$ defines a discriminator on each BAO in $\mathcal{K}$. The following is almost immediate from Givant's results [15, theorem 2.2, lemma 2.3].

Proposition 2.1. *If $\mathcal{K}$ is a discriminator class of BAOs with $\mathbf{Pu}\,\mathcal{K} \subseteq \mathbf{S}\,\mathcal{K}$, then $\mathbf{S}\,\mathbf{P}\,\mathcal{K}$ is a variety whose class of subdirectly irreducible members is $\mathbf{S}\,\mathcal{K}$.*

The dual $(n+1)$-ary relation symbol for an n-ary operator symbol $f \in L$ will be written R_f, and we write L^a for the similarity type consisting of these relation symbols. In this context, a *'structure'* will usually mean an L^a-structure. We write $\mathcal{A}_+$ for the *canonical structure* of an L-BAO $\mathcal{A}$; it consists of the set of all ultrafilters of (the boolean reduct of) $\mathcal{A}$, made into an L^a-structure via $\mathcal{A}_+ \models R_f(\mu_1, \ldots, \mu_n, \nu)$ iff $f(a_1, \ldots, a_n) \in \nu$ whenever $a_1 \in \mu_1, \ldots, a_n \in \mu_n$. We write S^+ for the complex algebra over the structure S; it consists of the set of all subsets of S, made into an L-BAO by defining $f(X_1, \ldots, X_n)$ to be the set of all y in S such that $S \models R_f(x_1, \ldots, x_n, y)$ for some $x_1 \in X_1, \ldots, x_n \in X_n$. The *canonical extension* $(\mathcal{A}_+)^+$ of a BAO $\mathcal{A}$ will be denoted by $\mathcal{A}^\sigma$; up to isomorphism, this is the 'perfect extension' of $\mathcal{A}$ defined by Jónsson and Tarski in [37]. A class of BAOs is said to be *canonical* if it is closed under taking canonical extensions. For a class $\mathcal{C}$ of structures, we write $\mathcal{C}^+$ for $\{S^+ : S \in \mathcal{C}\}$, and $\mathrm{Var}\,\mathcal{C}$ for the smallest variety containing $\mathcal{C}^+$; this is called the *variety generated by* $\mathcal{C}$. A variety of the form $\mathrm{Var}\,\mathcal{C}$ for an elementary class $\mathcal{C}$ is said to be *elementarily generated.* For a variety $\mathcal{V}$ of BAOs, we write $\mathrm{Cst}\,\mathcal{V} = \{\mathcal{A}_+ : \mathcal{A} \in \mathcal{V}\}$, and $\mathrm{Str}\,\mathcal{V} = \{S : S^+ \in \mathcal{V}\}$.

If S, T are L^a-structures, a map $\theta : S \to T$ is called a *bounded morphism* if for all n-ary operator symbols $f \in L$ and all $a \in S$, $b_1, \ldots, b_n \in T$, we have $T \models R_f(b_1, \ldots, b_n, \theta(a))$ iff there are $a_1, \ldots, a_n \in S$ with $S \models R_f(a_1, \ldots, a_n, a)$ and $\theta(a_1) = b_1, \ldots, \theta(a_n) = b_n$. If S is a substructure of T and the inclusion map from S to T is a bounded morphism, then S is called an *inner substructure* of T. If S_i $(i \in I)$ are pairwise disjoint inner substructures of T with $\bigcup_{i \in I} S_i = T$, we say that T is the *disjoint union* of the S_i, and write $T = \sum_{i \in I} S_i$. $\mathbf{Ud}\,\mathcal{C}$ will denote the closure under disjoint union of a class $\mathcal{C}$ of structures.

2.2. Canonical structures of products. The following is the specialisation to BAOs of a result proved in [14], with the main argument, concerning Stone spaces, going back to [12]. It is not so hard to give a proof in the BAO case, so we will do so to make our paper more self-contained (we will also use parts of the proof later on).

THEOREM 2.2. *If $\mathcal{K}$ is a canonical class of BAOs that is closed under ultraproducts, then $\mathbf{P}\,\mathcal{K}$ is also canonical.*

Notation. Throughout this subsection, let I be a non-empty set and let $\mathcal{A}_i$ $(i \in I)$ be a collection of similar BAOs. Write $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i$ and $S = \mathcal{A}_+$. Let $\mathrm{Spec}\, I$ denote the set of ultrafilters on I. For $X \subseteq I$ define $1_X \in \mathcal{A}$ by

$$(1_X)_i = \begin{cases} 1, & \text{if } i \in X, \\ 0, & \text{otherwise.} \end{cases}$$

Define the *support set* $\sigma(a)$ of $a = \langle a_i : i \in I \rangle \in \mathcal{A}$ to be $\sigma(a) = \{i \in I : a_i \neq 0\}$. Finally, for $\mu \in \mathcal{A}_+$, define $\sigma(\mu) = \{\sigma(a) : a \in \mu\}$.

It is clear that $\sigma(1_X) = X$ and $\sigma(\mu) = \{X \subseteq I : 1_X \in \mu\}$.

LEMMA 2.3. *For each $\mu \in \mathcal{A}_+$, $\sigma(\mu)$ is an ultrafilter on I.*

Proof. Clearly, $1 \in \mu$ and $\sigma(1) = I$, so $I \in \sigma(\mu)$. If $a \in \mu$ and $\sigma(a) \subseteq X \subseteq I$, then $1_X \geq a$ so $1_X \in \mu$ and $X = \sigma(1_X) \in \sigma(\mu)$. If $a, b \in \mu$ then $\sigma(a) \cap \sigma(b) \supseteq \sigma(a \cdot b) \in \sigma(\mu)$, so $\sigma(\mu)$ is closed under finite intersection. Finally, for any $X \subseteq I$, we have $X \in \sigma(\mu)$ iff $1_X \in \mu$, iff $1_{I \setminus X} = -1_X \notin \mu$, iff $I \setminus X \notin \sigma(\mu)$. So $\sigma(\mu)$ is an ultrafilter on I. □

Let $D \in \operatorname{Spec} I$. The map $a \mapsto a/D$ is a surjective homomorphism : $\mathcal{A} \to \prod_D \mathcal{A}_i$. By duality (see [2, theorem 5.47]), its inverse yields an injective bounded morphism $\nu_D : (\prod_D \mathcal{A}_i)_+ \to \mathcal{A}_+$. Write $\operatorname{rng} \nu_D$ for its range.

Lemma 2.4. $\operatorname{rng}(\nu_D) = \{\mu \in \mathcal{A}_+ : \sigma(\mu) = D\}$.

Proof. Let $f \in (\prod_D \mathcal{A}_i)_+$. If $a \in \nu_D(f)$ then $a/D \in f$, so $\prod_D \mathcal{A}_i \models a/D \neq 0$. This implies that $\sigma(a) \in D$. This holds for all such a; hence, by lemma 2.3, $\sigma(\nu_D(f)) = D$.

Conversely, if $\mu \in \mathcal{A}_+$ and $\sigma(\mu) = D$, the set $f = \{a/D : a \in \mu\}$ is easily seen to be an ultrafilter of $\prod_D \mathcal{A}_i$ with $\nu_D(f) = \mu$. □

Theorem 2.5. *For any similar BAOs $\mathcal{A}_i$ $(i \in I)$, we have*

$$\begin{aligned} \Big(\prod_{i\in I} \mathcal{A}_i\Big)_+ &\cong \sum_{D\in\operatorname{Spec} I} \Big(\Big(\prod_D \mathcal{A}_i\Big)_+\Big), \\ \Big(\prod_{i\in I} \mathcal{A}_i\Big)^\sigma &\cong \prod_{D\in\operatorname{Spec} I} \Big(\Big(\prod_D \mathcal{A}_i\Big)^\sigma\Big). \end{aligned}$$

Proof. Each $\operatorname{rng}(\nu_D)$ is (the domain of) an inner substructure of $\mathcal{A}_+$. By lemmas 2.3 and 2.4, the ranges of the ν_D for distinct D are pairwise disjoint, and $\bigcup_{D\in\operatorname{Spec} I} \operatorname{rng}(\nu_D) = \mathcal{A}_+$. So $\mathcal{A}_+ = \sum_{D\in\operatorname{Spec} I} \operatorname{rng}(\nu_D) \cong \sum_{D\in\operatorname{Spec} I}(\prod_D \mathcal{A}_i)_+$, proving the first line. The second line follows by duality (see [2, theorem 5.48]). □

Proof of theorem 2.2. Let $\prod_{i\in I} \mathcal{A}_i \in \mathbf{P}\,\mathcal{K}$ be given, with the $\mathcal{A}_i$ in $\mathcal{K}$. By our assumptions on $\mathcal{K}$, for each ultrafilter D on I, $\prod_D \mathcal{A}_i \in \mathcal{K}$, and so $(\prod_D \mathcal{A}_i)^\sigma \in \mathcal{K}$. So $\prod_{D\in\operatorname{Spec} I}((\prod_D \mathcal{A}_i)^\sigma) \in \mathbf{P}\,\mathcal{K}$. By the theorem, this is isomorphic to $(\prod_{i\in I} \mathcal{A}_i)^\sigma$ which is therefore in $\mathbf{P}\,\mathcal{K}$ as required (recall that $\mathbf{P}$ denotes closure under isomorphic copies of products). □

2.3. A non-elementarily generated canonical variety. We consider BAOs in signature $L = \{+, -, 0, 1, f, d\}$, where f and d are unary operator symbols. We will use d as a discriminator.

Definition 2.6. For an L-BAO $\mathcal{A}$, an element $a \in \mathcal{A}$ is said to be *independent* if $a \cdot f(a) = 0$. We write $\chi(\mathcal{A})$ for the least $n < \omega$ such that there are independent $a_0, \ldots, a_{n-1} \in \mathcal{A}$ with $\sum_{i<n} a_i = 1$, and ∞ if there is no such n.

Note that if $a_0, \ldots, a_{n-1}$ are as above, and we let $b_i = a_i \cdot \prod_{j<i} -a_j$, then $b_0, \ldots, b_{n-1}$ have the same properties and are pairwise disjoint. So the terminology in definition 2.6 is consistent with standard graph theory, if we regard a graph G as a structure for $L^a = \{R_f, R_d\}$ by interpreting R_f as the graph edge relation (and R_d as the universal relation $G \times G$). For instance, $\chi(G^+)$ coincides with the chromatic number of G. Also note that if $\mathcal{A}$ is degenerate ($|\mathcal{A}| = 1$) then $0 = 1$ is an independent element, so $\chi(\mathcal{A}) = 1$.

Lemma 2.7. *Let $\mathcal{A}, \mathcal{B}$ be L-BAOs.*

1. *If there is a homomorphism $h : \mathcal{A} \to \mathcal{B}$, then $\chi(\mathcal{B}) \le \chi(\mathcal{A})$.*
2. *If $\mathcal{A} \subseteq \mathcal{B}$ then $\chi(\mathcal{B}) \le \chi(\mathcal{A})$.*
3. *If $\chi(\mathcal{A}) = \infty$ then $\chi(\mathcal{A}^\sigma) = \infty$.*

Proof.

1. For any finite n, if there are independent $a_0, \ldots, a_{n-1} \in \mathcal{A}$ with $\sum_{i<n} a_i = 1$, then $h(a_0), \ldots, h(a_{n-1}) \in \mathcal{B}$ have the same properties. So $\chi(\mathcal{B}) \le \chi(\mathcal{A})$.
2. This follows from part 1.
3. Assume that $\chi(\mathcal{A}) = \infty$. Let I be the ideal of the boolean reduct of $\mathcal{A}$ generated by the independent elements of $\mathcal{A}$. Since in this algebra the join of finitely many independent elements is never equal to 1, I is a proper ideal. Extend it to a maximal ideal I^*, and let $\mu = \mathcal{A} \setminus I^* \in \mathcal{A}_+$. We claim that $f(a) \in \mu$ for all $a \in \mu$. For if not, take $a \in \mu$ with $f(a) \notin \mu$. So $-f(a) \in \mu$. Let $b = a \cdot -f(a) \in \mu$. Then $b \le a$, so $f(b) \le f(a)$ by monotony of f. So $b \cdot f(b) \le b \cdot f(a) = a \cdot -f(a) \cdot f(a) = 0$. This means that b is independent, and hence in $I \subseteq \mathcal{A} \setminus \mu$ by definition of I, μ. Since we know that $b \in \mu$, this is a contradiction, and proves the claim.

   By the claim, μ is an R_f-reflexive element of $\mathcal{A}_+$. No element of $\mathcal{A}^\sigma$ containing μ can be independent. But whenever $\sum_{i<n} a_i = 1$ in $\mathcal{A}^\sigma$, we must have $\mu \in a_i$ for some $i < n$, and so a_i is not independent. So we see that $\chi(\mathcal{A}^\sigma) = \infty$.

□

By part 3 of the lemma, if $\chi(\mathcal{A}) = \infty$ then $\chi(\mathcal{A}^\sigma) = \chi(\mathcal{A})$. We remark that this can fail if $\mathcal{A}$ has finite chromatic number: [32, theorem 4.2] constructs, for each k, l with $2 \le l \le k < \omega$, an L-BAO $\mathcal{A}$ with $\chi(\mathcal{A}) = k$ and $\chi(\mathcal{A}^\sigma) = l$.

For each finite $n \ge 2$ fix a finite graph G_n with chromatic number $> n$ and no cycles of length $< n$. This exists by Erdős's famous result [9] (see [8, chapter 11] for a recent presentation). We can assume that if $n < m$ then $|G_n| < |G_m|$. For integers $n, m \ge 1$, let

$$\sigma[n, m] \;=\; \exists_{\ge n} x(x = x) \to \neg \operatorname*{\exists}_{i<m} x_i \Big(\sum_{i<m} x_i = 1 \wedge \bigwedge_{i<m} (x_i \cdot f(x_i) = 0) \Big),$$

saying that if $\mathcal{A}$ has at least n elements then $\chi(\mathcal{A}) > m$. Define

$$\begin{aligned} \Sigma \;&=\; \{\sigma[2,2]\} \cup \{\sigma[2^{|G_n|}, n] : n \ge 2\} \\ &\qquad \cup \{d(0) = 0 \wedge \forall x(x > 0 \to d(x) = 1)\}, \\ \mathcal{K} \;&=\; \{\mathcal{A} : \mathcal{A} \text{ is an } L\text{-BAO}, \mathcal{A} \models \Sigma\}. \end{aligned}$$

Note that each sentence in Σ is equivalent to a universal sentence; so $\mathcal{K} = \mathbf{S}\,\mathcal{K}$. As $\mathcal{K}$ is elementary, it is closed under ultraproducts.

LEMMA 2.8. *$\mathcal{K}$ is canonical.*

Proof. Let $\mathcal{A} \in \mathcal{K}$. If $\mathcal{A}$ is finite, then $\mathcal{A}^\sigma \cong \mathcal{A} \in \mathcal{K}$. If it is infinite, then $|\mathcal{A}| \geq 2^{|G_n|}$ for all $n \geq 2$, so as $\mathcal{A} \models \sigma[2^{|G_n|}, n]$, we have $\chi(\mathcal{A}) > n$ for all $n \geq 2$. Hence, $\chi(\mathcal{A}) = \infty$. By lemma 2.7(3), $\chi(\mathcal{A}^\sigma) = \infty$ as well; so certainly, $\mathcal{A}^\sigma \models \{\sigma[2,2]\} \cup \{\sigma[2^{|G_n|}, n] : n \geq 2\}$. It is easily seen that if d is a discriminator on $\mathcal{A}$ then it is on $\mathcal{A}^\sigma$ as well. So $\mathcal{A}^\sigma \in \mathcal{K}$. □

DEFINITION 2.9. Let $\mathcal{V} = \mathbf{S}\,\mathbf{P}\,\mathcal{K}$.

LEMMA 2.10. *$\mathcal{V}$ is a canonical variety.*

Proof. $\mathcal{K}$ is closed under ultraproducts, so by lemma 2.8 and theorem 2.2, $\mathbf{P}\,\mathcal{K}$ is canonical. By (e.g.) [29, theorem 2.71], if $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A}^\sigma \subseteq \mathcal{B}^\sigma$ up to isomorphism. So $\mathcal{V} = \mathbf{S}\,\mathbf{P}\,\mathcal{K}$ is also canonical. Since $\mathcal{K}$ is a discriminator class, it follows from proposition 2.1 that $\mathcal{V}$ is a variety. □

LEMMA 2.11. *$\chi(\mathcal{A}) > 2$ for each non-degenerate $\mathcal{A} \in \mathcal{V}$.*

Proof. The result holds for each non-degenerate $\mathcal{A} \in \mathcal{K}$, since $\mathcal{A} \models \sigma[2,2]$. Assume that $\mathcal{A}_i \in \mathcal{K}$ ($i \in I$) are not all degenerate and $\chi(\prod_{i\in I} \mathcal{A}_i) \leq 2$. Noting that for each $i \in I$, the projection from $\prod_{j\in I} \mathcal{A}_j$ to $\mathcal{A}_i$ is a homomorphism, by lemma 2.7(1) we must have $\chi(\mathcal{A}_i) \leq 2$ for any $i \in I$ with $\mathcal{A}_i$ non-degenerate, a contradiction. So the result holds for $\mathbf{P}\,\mathcal{K}$. Finally, if $\mathcal{A}$ is non-degenerate, $\mathcal{A} \subseteq \mathcal{B} \in \mathbf{P}\,\mathcal{K}$, and $\chi(\mathcal{A}) \leq 2$, then $\chi(\mathcal{B}) \leq 2$ by lemma 2.7(2), contradicting the result for $\mathbf{P}\,\mathcal{K}$. So the result holds for $\mathcal{V} = \mathbf{S}\,\mathbf{P}\,\mathcal{K}$, as required. □

As indicated above, we regard each G_n as a structure for $L^a = \{R_f, R_d\}$ by interpreting R_f as the graph edge relation and R_d as the universal relation $G_n \times G_n$. Then G_n^+ is an L-BAO with $2^{|G_n|}$ elements. Also, $\chi(G_n^+)$ is equal to the chromatic number of G_n, and hence $\chi(G_n^+) > n$.

LEMMA 2.12. *For each $n \geq 2$, $G_n \in \operatorname{Cst}\mathcal{V}$ up to isomorphism.*

Proof. Let $n \geq 2$. We first show that $G_n^+ \in \mathcal{K}$. G_n^+ has chromatic number $> n \geq 2$, so $G_n^+ \models \sigma[2,2]$. Let $m \geq 2$; we check that $G_n^+ \models \sigma[2^{|G_m|}, m]$. If $m \leq n$ then the consequent of $\sigma[2^{|G_m|}, m]$ is true in G_n^+, since $\chi(G_n^+) > n \geq m$. If $m > n$, then the antecedent of $\sigma[2^{|G_m|}, m]$ is false in G_n^+, since this algebra has exactly $2^{|G_n|}$ elements, and $|G_m| > |G_n|$. So $G_n^+ \models \sigma[2^{|G_m|}, m]$ in each case. Certainly, d is a discriminator in G_n^+. So $G_n^+ \models \Sigma$. By its definition, G_n^+ is an L-BAO. So $G_n^+ \in \mathcal{K}$ for all $n \geq 2$.

For each $n \geq 2$, $G_n^+ \in \mathcal{K} \subseteq \mathcal{V}$, so as G_n is finite, $G_n \cong (G_n^+)_+ \in \operatorname{Cst}\mathcal{V}$. □

Let G be a non-principal ultraproduct of the G_n over $\omega \setminus 2$.

Lemma 2.13. $G \notin \mathrm{Str}\,\mathcal{V}$.

Proof. For each k, only finitely many of the G_l have any cycles of length k. Now by Łoś' theorem, any first-order sentence true in G is also true in infinitely many of the G_l. Since the property of having a cycle of length k is expressible by a first-order sentence, it follows that G has no cycles. So G is 2-colourable, and hence $\chi(G^+) \leq 2$. Certainly, G^+ is non-degenerate. By lemma 2.11, $G^+ \notin \mathcal{V}$. □

Remark 2.14. This lemma can also be proved by using proposition 2.1, from which it follows that if $G^+ \in \mathcal{V}$ then $G^+ \in \mathcal{K}$. But G is infinite, so G^+ does not validate any of the axioms $\sigma[2^{|G_n|}, n]$ $(n \geq 2)$. This approach obviates the need for lemma 2.11 and the axiom $\sigma[2,2]$ in the definition of $\mathcal{K}$.

Lemma 2.15. *The universal theory of $\mathcal{V}$ has the finite algebra property: i.e., any universal L-sentence not valid in $\mathcal{V}$ fails in a finite algebra in $\mathcal{V}$.*

Proof. Let ρ be a universal L-sentence such that $\mathcal{A} \not\models \rho$ for some $\mathcal{A} \in \mathcal{V}$. Then $\mathcal{A}^\sigma \not\models \rho$ as ρ is preserved by subalgebras. For the same reason, since $\mathcal{V} = \mathbf{S}\,\mathbf{P}\,\mathcal{K}$ we can assume that $\mathcal{A} \in \mathbf{P}\,\mathcal{K}$. Hence by theorem 2.5, $\mathcal{A}_+ \cong T$, where T is a disjoint union of structures of the form $\mathcal{B}_+$ with $\mathcal{B} \in \mathcal{K}$. Then $\mathcal{A}^\sigma \cong T^+$, so $T^+ \not\models \rho$ and $T^+ \in \mathcal{V}$ as $\mathcal{V}$ is a canonical variety.

Assume the matrix (quantifier-free part) of ρ is in conjunctive normal form. Distribute the universal quantifiers across the conjunctions. One of the resulting conjuncts τ has $T^+ \not\models \tau$. This τ is a universal sentence whose matrix is a disjunction. Each disjunct of τ is either an equation which can be taken in the form $t = 0$, in which case we say that the term t is *positive in* τ, or the negation $u \neq 0$ of an equation, in which case u is *negative in* τ. Pick a valuation h mapping the variables of τ into T^+ so that $(T^+, h) \not\models t = 0$ when t is positive and $(T^+, h) \models u = 0$ when u is negative. Let $h(s) \in T^+$ denote the value of the term s in (T^+, h).

For each of the finitely many positive terms t in τ pick some $a^t \in T$ with $a^t \in h(t)$, and some $\mathcal{B}^t \in \mathcal{K}$ such that $a^t \in \mathcal{B}^t_+ \subseteq T$. Let $\mathcal{A}^t = (\mathcal{B}^t_+)^+ = (\mathcal{B}^t)^\sigma$. Then $\mathcal{A}^t \in \mathcal{K}$ as $\mathcal{K}$ is canonical. The function $f^t(X) = X \cap \mathcal{B}^t_+$ is a surjective homomorphism $T^+ \to \mathcal{A}^t$ as $\mathcal{B}^t_+$ is an inner substructure of T.

Let $h^t = f^t \circ h$. Then $h^t(s) = f^t(h(s))$ for all terms s as f^t is a homomorphism. Thus $h^t(u) = 0$ for all negative u, and $a^t \in h(t) \cap \mathcal{B}^t_+ = h^t(t)$, so $h^t(t) \neq 0$. Then if $t_1, \ldots t_n$ are all the positive terms in τ, put $\mathcal{B} = \mathcal{A}^{t_1} \times \cdots \times \mathcal{A}^{t_n}$, and $h'(v) = (h^{t_1}(v), \ldots, h^{t_n}(v))$ for all variables occurring in τ. Then $h'(s) = (h^{t_1}(s), \ldots, h^{t_n}(s))$ for all terms in these variables, so $h'(u) = 0$ for all negative u in τ and $h'(t_i) \neq 0$ for all positive t_i. This shows $\mathcal{B} \not\models \tau$, and hence ρ fails in $\mathcal{B}$.

But $\mathcal{B} \in \mathcal{V}$, since each $\mathcal{A}^t \in \mathcal{K}$. If each $\mathcal{A}^t$ is finite, then $\mathcal{B}$ is the desired finite falsifying algebra in $\mathcal{V}$ for ρ, and the proof is finished. If however any $\mathcal{A}^t$ is infinite, we use filtration to collapse it to a finite algebra that still has the properties needed of $\mathcal{A}^t$.

To do this, let Z be the finite set of all subterms of terms occurring in τ. Define an equivalence relation $\sim$ on $\mathcal{B}^t_+$ by $a \sim b$ iff $a \in h^t(s) \iff b \in h^t(s)$ for all $s \in Z$. Define the relations R_f, R_d existentially on the quotient $S_t = (\mathcal{B}^t_+)/{\sim}$, by $S_t \models R_f(a/{\sim}, b/{\sim})$ iff $\mathcal{B}^t_+ \models R_f(a', b')$ for some $a', b' \in \mathcal{B}^t_+$ with $a' \sim a$ and $b' \sim b$, and similarly for R_d. For each variable v occurring in τ, define $h^t_\sim(v) = \{a/{\sim} : a \in h^t(v)\} \in S^+_t$. It is now easy to check by induction on formation of terms $s \in Z$ that $h^t_\sim(s) = \{a/{\sim} : a \in h^t(s)\}$. Hence, $h^t_\sim(u) = 0$ in S^+_t for all negative u in τ, while $a^t/{\sim} \in h^t_\sim(t)$, so $h^t_\sim(t) \neq 0$.

Clearly, S^+_t is finite. It remains to check that $S^+_t \in \mathcal{V}$. Now since $\mathcal{A}^t$ is infinite, so are $\mathcal{B}^t_+$ and $\mathcal{B}^t$. Hence, by the proof of lemma 2.7(3), $\mathcal{B}^t_+$ contains an R_f-reflexive point, say a. Then $a/{\sim} \in S_t$ is also reflexive. It follows that $\chi(S^+_t) = \infty$, so S^+_t satisfies the consequents of all the axioms $\sigma[n, m]$ defining $\mathcal{K}$. Also, since $(\mathcal{B}^t_+)^+ \in \mathcal{K}$, the property of d being a discriminator on $(\mathcal{B}^t_+)^+$ is inherited by S^+_t, since this property means that R_d is the universal relation on each structure. So in fact $S^+_t \in \mathcal{K}$. Thus we can replace the factor $\mathcal{A}^t$ of $\mathcal{B}$ by the finite $\mathcal{V}$-algebra S^+_t in each case that $\mathcal{A}^t$ is infinite, to complete the construction as desired. □

Corollary 2.16. *One may choose the G_n $(n \geq 2)$ so that the universal theory of $\mathcal{V}$ is decidable.*

Proof. Fix a recursive enumeration of all isomorphism types of finite graphs, in order of their cardinality. If G_m $(2 \leq m < n)$ have been defined, define G_n to be the first graph in the enumeration with chromatic number $> n$, no cycles of length $< n$, and with $|G_n| > |G_m|$ for all m with $2 \leq m < n$. This yields a recursive enumeration of the axioms defining $\mathcal{K}$; from this one may easily obtain a recursive enumeration of the equational theory of $\mathcal{K}$, which by lemma 2.10 axiomatises $\mathcal{V}$. Hence, the universal theory of $\mathcal{V}$ is also recursively enumerable.

On the other side, a finite non-degenerate L-BAO $\mathcal{A}$ is in $\mathcal{K}$ iff d is a discriminator on it, and its chromatic number is > 2 and also $> n$ for all $n \geq 2$ such that $2^{|G_n|} \leq |\mathcal{A}|$. There are finitely many such n, so this constitutes an algorithm to decide whether $\mathcal{A} \in \mathcal{K}$. Now by [1], any finite $\mathcal{B} \in \mathcal{V}$ has a subdirect decomposition of the form $\mathcal{B} \subseteq \prod_{i \in I} \mathcal{A}_i$, where each $\mathcal{A}_i$ is subdirectly irreducible and a homomorphic image of $\mathcal{B}$. So for each i, $|\mathcal{A}_i| \leq |\mathcal{B}|$ and $\mathcal{A}_i \in \mathcal{V}$; and by proposition 2.1, $\mathcal{A}_i \in \mathbf{S}\,\mathcal{K} = \mathcal{K}$. For each nonzero $b \in \mathcal{B}$, choose some $i_b \in I$ such that the projection of $\mathcal{B}$ onto $\mathcal{A}_{i_b}$ takes b to a non-zero element. Then $\mathcal{B} \subseteq \prod_{b \in \mathcal{B}} \mathcal{A}_{i_b}$, so we can suppose without loss of generality that $|I| \leq |\mathcal{B}|$. Hence we may recursively enumerate the isomorphism types of finite algebras in $\mathcal{V}$ by enumerating all subalgebras of finite products of finite algebras in $\mathcal{K}$. So, using lemma 2.15, we may enumerate all universal sentences not valid in $\mathcal{V}$ by simultaneously enumerating all universal L-sentences α and isomorphism types of finite L-BAOs $\mathcal{B} \in \mathcal{V}$, checking whether $\mathcal{B} \models \alpha$, and printing out α if not.

Any universal L-sentence will occur in just one of these two enumerations. We can use this in the usual way to decide the universal theory of $\mathcal{V}$. □

Proposition 2.17. *A variety $\mathcal{V}$ of BAOs is elementarily generated iff it is canonical and there is an elementary class $\mathcal{S}$ of structures satisfying* $\mathrm{Cst}\,\mathcal{V} \subseteq \mathcal{S} \subseteq \mathrm{Str}\,\mathcal{V}$.

Proof. Assume that $\mathcal{V}$ is canonical and there is such an $\mathcal{S}$. Then $\mathcal{S}^+ \subseteq \mathcal{V}$, so the variety $\mathrm{Var}\,\mathcal{S}$ generated by $\mathcal{S}$ is contained in $\mathcal{V}$. But by canonicity, $\mathcal{V} \subseteq \mathbf{S}(\mathrm{Cst}\,\mathcal{V})^+ \subseteq \mathbf{S}\,\mathcal{S}^+ \subseteq \mathrm{Var}\,\mathcal{S}$.

Conversely, if $\mathcal{V} = \mathrm{Var}\,\mathcal{C}$ for some class $\mathcal{C}$ of structures that is closed under ultraproducts, then by [19, theorem 4.12], $\mathcal{S} = \mathbf{Ru}\,\mathbf{S}\,\mathbf{H}\,\mathbf{Ud}\,\mathcal{C}$ is as required. By [17, theorem 3.6.7], $\mathcal{V}$ is canonical. □

Theorem 2.18. *There is a canonical variety of BAOs with the finite algebra property and decidable universal theory, that is not elementarily generated.*

Proof. The $\mathcal{V}$ of definition 2.9 is a canonical variety and can be taken to have the other two positive properties, by lemmas 2.10 and 2.15 and corollary 2.16. If it were determined by an elementary class of frames, proposition 2.17 shows that there would be an elementary class $\mathcal{S}$ with $\mathrm{Cst}\,\mathcal{V} \subseteq \mathcal{S} \subseteq \mathrm{Str}\,\mathcal{V}$. Hence, any ultraproduct of structures in $\mathrm{Cst}\,\mathcal{V}$ would lie in $\mathrm{Str}\,\mathcal{V}$. But by lemma 2.12, up to isomorphism we have $G_n \in \mathrm{Cst}\,\mathcal{V}$ (all $n \geq 2$); and by lemma 2.13, $G \notin \mathrm{Str}\,\mathcal{V}$. □

If we do not desire decidability, we can strengthen this result.

Theorem 2.19. *There are $2^{\aleph_0}$ distinct canonical varieties of L-BAOs with the finite algebra property and not elementarily generated.*

Proof. Let c_n be the chromatic number of G_n (for each $n \geq 2$). We may assume that if $2 \leq m < n$ then $|G_m| < |G_n|$ and $c_m < c_n$. For $X \subseteq \omega \setminus 2$ let

$$\begin{aligned} \Sigma_X &= \{\sigma[2^{|G_n|}, c_n - 1] : n \in X\} \cup \{\sigma[2^{|G_n|}, c_n] : n \geq 2,\ n \notin X\} \\ &\qquad \cup \{\sigma[2,2]\} \cup \{d(0) = 0 \wedge \forall x(x > 0 \to d(x) = 1)\}, \\ \mathcal{K}_X &= \{L\text{-BAOs } \mathcal{A} : \mathcal{A} \models \Sigma_X\}, \\ \mathcal{V}_X &= \mathbf{S}\,\mathbf{P}\,\mathcal{K}_X. \end{aligned}$$

Claim. For each $n \geq 2$ we have $G_n^+ \in \mathcal{V}_X$ iff $n \in X$.
Proof of claim. Assume that $n \in X$. Certainly, $\chi(G_n^+) = c_n > c_n - 1 \geq 2$, so $G_n^+ \models \sigma[2,2] \wedge \sigma[2^{|G_n|}, c_n - 1]$. For $m \geq 2$, if $m < n$ then $c_n > c_m$,

so $G_n^+ \models \sigma[2^{|G_m|}, c_m - 1] \wedge \sigma[2^{|G_m|}, c_m]$. If $m > n$ then the antecedents of $\sigma[2^{|G_m|}, c_m - 1]$ and $\sigma[2^{|G_m|}, c_m]$ fail in G_n^+, so both sentences are true in G_n^+. Hence $G_n^+ \in \mathcal{K}_X \subseteq \mathcal{V}_X$.

Conversely, assume that $n \geq 2$ and $G_n^+ \in \mathcal{V}_X$. G_n^+ is subdirectly irreducible since d is a discriminator on it. We know that $\mathbf{Pu}\,\mathcal{K}_X = \mathcal{K}_X \subseteq \mathbf{S}\,\mathcal{K}_X$. By proposition 2.1, $G_n^+ \in \mathbf{S}\,\mathcal{K}_X$. Σ_X is a universal theory; so $\mathbf{S}\,\mathcal{K}_X = \mathcal{K}_X$, $G_n^+ \in \mathcal{K}_X$, and $G_n^+ \models \Sigma_X$. Hence, if $n \notin X$, we must have $G_n^+ \models \sigma[2^{|G_n|}, c_n]$; but $|G_n^+| \geq 2^{|G_n|}$ and $\chi(G_n^+) \not> c_n$, a contradiction. So $n \in X$, proving the claim.

Using the claim, earlier proofs now show that for any infinite $X \subseteq \omega \setminus 2$, $\mathcal{V}_X$ is canonical and has the finite algebra property, but is not elementarily generated. (We need X infinite in order that $\mathcal{V}_X$ contain infinitely many algebras G_n^+, so that the ultraproduct part of the proof of theorem 2.18 goes through.) By the claim, if $X, Y \subseteq \omega \setminus 2$ are distinct then $\mathcal{V}_X \neq \mathcal{V}_Y$. So $\{\mathcal{V}_X : X \subseteq \omega \setminus 2,\ X \text{ infinite}\}$ is a class of $2^{\aleph_0}$ varieties with the required properties. $\square$

There are only countably many algorithms, so not all $\mathcal{V}_X$ can have decidable universal theory.

§3. The modal approach. We now briefly give a similar argument in purely modal terms. We assume some familiarity with modal logic: modal languages and their semantics, basic frame theory (including bounded morphisms and inner subframes), normal modal logics and notions pertaining to them such as soundness, completeness, canonicity, and the finite model property, and the Jankov–Fine formula encoding the modal diagram of a frame. All the material we need can be found in [2] and [5].

We use a modal language with two boxes, written $\Box, \mathsf{A}$. We will write $R_\Box, R_\mathsf{A}$ for their accessibility relations, and $\Diamond, \mathsf{E}$ for the corresponding diamonds. The operator A is intended as a global or universal modality (see [2]); frames $F = (W, R_\Box, R_\mathsf{A})$ on which, indeed, $R_\mathsf{A} = W \times W$ will be called *standard*. For $F = (W, R_\Box, R_\mathsf{A})$, we will write $|F|$ for $|W|$.

A *colouring* of a frame $F = (W, R_\Box, R_\mathsf{A})$ is a collection C of subsets of W such that $\bigcup C = W$ and $F \models \neg R_\Box(x, y)$ for all $x, y \in S$ and all $S \in C$. The *chromatic number* $\chi(F)$ of F is the least $m < \omega$ for which there exists a colouring of F of cardinality m; we set $\chi(F) = \infty$ if F has no finite colouring. Note that although colourings need not partition the domain of the frame, any finite colouring can be refined to one that does. So if we consider a graph $G = (V, E)$ as a frame $F = (V, E, V \times V)$, the chromatic number of F coincides with the chromatic number of G as usually defined in graph theory (as in §1).

$|F|$ and $\chi(F)$ are two 'largeness notions' for frames F. They are to an extent modally definable:

LEMMA 3.1. *Let $F = (W, R_\Box, R_\mathsf{A})$ be a standard frame, let $n, m < \omega$, and let $p_0, \ldots, p_{n-1}$, $q_0, \ldots, q_{m-1}$ be distinct propositional variables.*

1. *The formula $\bigwedge_{i<n} \mathsf{E}(p_i \wedge \bigwedge_{j<i} \neg p_j)$ is satisfiable in F iff $|F| \geq n$.*
2. *The formula $\mathsf{A} \bigvee_{i<m}(q_i \wedge \Box\neg q_i)$ is satisfiable in F iff $\chi(F) \leq m$.*

Proof. For the first part, assume that $\bigwedge_{i<n} \mathsf{E}(p_i \wedge \bigwedge_{j<i} \neg p_j)$ is satisfiable in F under some assignment h of the variables. For each $i < n$, pick $w_i \in W$ with $(F, h), w_i \models p_i \wedge \bigwedge_{j<i} \neg p_j$. The w_i must clearly be pairwise distinct; so $|F| \geq n$. Conversely, if $|F| \geq n$ then assigning $p_0, \ldots, p_{n-1}$ to distinct singletons in $\wp(W)$ will satisfy the formula.

Assume now, in order to prove part 2 of the lemma, that $\mathsf{A} \bigvee_{i<m}(q_i \wedge \Box\neg q_i)$ is satisfiable in F under some assignment h. For each $i < m$, let $S_i = \{w \in W : (F, h), w \models q_i \wedge \Box\neg q_i\}$. Then the S_i witness that $\chi(F) \leq m$. Conversely, assume that there are sets $S_i \subseteq W$ $(i < m)$ with union W and such that $F \models \neg R_\Box(x, y)$ for all $x, y \in S_i$ and $i < m$. Assign q_i to S_i (each $i < m$) and observe that the formula is now true at any world of F. □

DEFINITION 3.2. For $n, m < \omega$ and distinct propositional variables $p_0, \ldots, p_{n-1}$, $q_0, \ldots, q_{m-1}$, let

$$\alpha[n, m] = \left(\bigwedge_{i<n} \mathsf{E}(p_i \wedge \bigwedge_{j<i} \neg p_j) \right) \to \mathsf{E} \bigwedge_{i<m} (\Box q_i \to q_i).$$

LEMMA 3.3. *Let F be a standard frame. Then $\alpha[n, m]$ is valid in F iff (if $|F| \geq n$ then $\chi(F) > m$).*

Proof. The formula $\alpha[n, m]$ is not valid in F iff $\bigwedge_{i<n} \mathsf{E}(p_i \wedge \bigwedge_{j<i} \neg p_j)$ and $\mathsf{A} \bigvee_{i<m}(\Box q_i \wedge \neg q_i)$ are both satisfiable in F (since the truth of these formulas does not depend on the evaluation point). By lemma 3.1, this is iff $|F| \geq n$ and $\chi(F) \leq m$. □

For each $n < \omega$ let G_n be a finite graph with chromatic number $> n$ and no cycles of length $< n$ (see Erdős's paper [9] for their existence). We write $|G_n|$ for the number of nodes of G_n. We may suppose that $|G_0| < |G_1| < \cdots$.

DEFINITION 3.4. Let EG (standing for 'Erdős graphs') be the normal modal logic (in the modal language above) axiomatised by:

1. all propositional tautologies,
2. normality: $\Box(p \to q) \to (\Box p \to \Box q)$, and $\mathsf{A}(p \to q) \to (\mathsf{A}p \to \mathsf{A}q)$,
3. $\{\alpha[|G_n|, n] : n < \omega\}$,
4. the axioms $\mathsf{A}p \to \Box p$, $\mathsf{A}p \to p$, and $\mathsf{E}p \to \mathsf{AE}p$, expressing that A is a global or universal modality.

Its derivation rules are modus ponens, universal generalisation for each of the two boxes, and uniform substitution (of variables by formulas).

The symmetry axiom $p \to \Box\Diamond p$ can be added if desired, but it is not needed.

LEMMA 3.5. *The logic* EG *is canonical.*

Proof. Fix a set $\mathcal{L}$ of propositional variables. Using formulas written with variables from $\mathcal{L}$, let $M = (K, h)$ be the canonical model of EG, with underlying frame K. We show that K is a frame for EG.

Let C be any R_{A}-cluster of K, regarded as a subframe of K. C is an inner subframe, so it suffices to check that C is a frame for EG; and since C is a standard frame we need not worry about the axioms dealing with the global modality. Thus it remains to verify that C validates the formulas $\alpha[|G_n|, n]$ for $n < \omega$. If C is finite, this is clear, as any valuation into C is definable in M, and the model M validates EG. So assume that C is infinite.

Claim. There is $\Gamma \in C$ with $K \models R_\Box(\Gamma, \Gamma)$.
Proof of claim. There is a similar argument in Hughes's paper [33]. Pick any $\Delta \in C$. It suffices to show that the set

$$\Gamma_0 = \{\Box\varphi \to \varphi : \varphi \text{ an } \mathcal{L}\text{-formula}\} \cup \{\delta : \mathsf{A}\delta \in \Delta\}$$

is EG-consistent; for any maximal consistent set Γ containing it will be $R_\Box$-reflexive and in C.

Assume for contradiction that Γ_0 is inconsistent. So by normality, there are $\mathsf{A}\delta \in \Delta$ and $\mathcal{L}$-formulas $\varphi_0, \ldots, \varphi_{m-1}$ for some $m < \omega$, such that $\mathsf{EG} \vdash \delta \to \neg \bigwedge_{i<m}(\Box\varphi_i \to \varphi_i)$. Applying universal generalisation and normality yields $\mathsf{EG} \vdash \mathsf{A}\delta \to \mathsf{A}\neg \bigwedge_{i<m}(\Box\varphi_i \to \varphi_i)$. Hence,

$$\mathsf{A}\neg \bigwedge_{i<m} (\Box\varphi_i \to \varphi_i) \in \Delta. \tag{1}$$

Now let $n = |G_m|$ and define formulas ψ_i $(i < n)$ as follows. Since C is infinite it is not hard to find distinct $\Gamma_0, \ldots, \Gamma_{n-1} \in C$ and formulas $\gamma_{ij} \in \Gamma_i \backslash \Gamma_j$ separating Γ_i from Γ_j. Let $\psi_i = \bigwedge_{j \neq i} \gamma_{ij}$. Then for all $i, j < n$, we have $\psi_i \in \Gamma_j$ iff $i = j$; in fact, we obtain $M, \Gamma_i \models \psi_i \wedge \bigwedge_{j<i} \neg\psi_j$ for each $i < n$.

Since $\Delta \in C$, we have $\bigwedge_{i<n} \mathsf{E}(\psi_i \wedge \bigwedge_{j<i} \neg\psi_j) \in \Delta$ by the truth lemma for M. But $\alpha[n, m]$ is an axiom of EG; so we obtain $\mathsf{E} \bigwedge_{i<m}(\Box\varphi_i \to \varphi_i) \in \Delta$. Taken with (1), this contradicts the consistency of Δ, and proves the claim.

Any frame with an $R_\Box$-reflexive point has chromatic number ∞, so by lemma 3.3 validates $\alpha[n, m]$ for all n, m. This, with the claim, implies that C is a frame for EG. Hence, K is also a frame for EG, as required. $\Box$

Lemma 3.6. EG *is not sound and complete for any elementary class of frames.*

Proof. Assume for contradiction that EG is sound and complete for some elementary class $\mathcal{K}$ of frames. Let $n < \omega$. We regard G_n as a standard frame for the modal type above by interpreting $R_\Box$ as the graph edge relation (and R_A as the universal relation $G_n \times G_n$). It can be checked using lemma 3.3 that G_n validates EG. Let ψ_n be (essentially) the Jankov–Fine formula of G_n (see, e.g., [2, §3.4] and [5, §9.4]):

$$\Big(\mathsf{A}\bigvee_{x\in G_n}\Big(x \wedge \bigwedge_{y\in G_n\setminus\{x\}} \neg y\Big)\Big) \wedge \Big(\bigwedge_{x\in G_n} \mathsf{E}x\Big) \wedge \mathsf{A}\bigwedge_{x\in G_n}\Big(\Diamond x \leftrightarrow \bigvee_{R_\Box(y,x)} y\Big),$$

where we regard each $x \in G_n$ as a propositional variable. Then ψ_n is satisfiable in G_n. So ψ_n is EG-consistent, and hence there is $F_n \in \mathcal{K}$ in which ψ_n is satisfiable. Since F_n validates EG, R_A defines an equivalence relation on it, each equivalence class being an inner subframe of F_n. The form of ψ_n now implies that there is an inner subframe $I_n \subseteq F_n$ and a surjective bounded morphism $m_n : I_n \to G_n$ (see, e.g., [2, lemma 3.20] for details).

Now consider the class $\mathcal{T}$ of structures of the form (A, B, m), where $A \in \mathcal{K}$, B is a standard frame disjoint from A, and $m \subseteq A \times B$ is a surjective bounded morphism from an inner subframe of A onto B. Since $\mathcal{K}$ is elementary, these statements are first-order expressible, and we can find a first-order theory T, say, containing first-order sentences that together axiomatise $\mathcal{T}$, and additional sentences stating that 'B' (above) has at least n elements for each finite n, $R_\Box$ is irreflexive and symmetric on B, and B has no $R_\Box$-cycles of length n for each finite n. Any finite subset of T has a model, namely, (F_n, G_n, m_n) for any large enough n. By compactness for first-order logic, we may take $(F, G, m) \models T$. Then $F \in \mathcal{K}$, so F is an EG-frame. The domain of m is an inner subframe of F, so also an EG-frame. G is a bounded morphic image of this, so is itself an EG-frame.

But $R_\Box$ is irreflexive and symmetric on G and has no cycles. Hence, $\chi(G) \leq 2$. Also, G is infinite and standard. By lemma 3.3, G does not validate any of the axioms $\alpha[|G_n|, n]$ of EG, and so is not an EG-frame. This contradiction completes the proof. □

Remark 3.7. The same argument shows that the variant of EG in which the axioms $\alpha[|G_n|, n]$ are replaced by any single axiom of the form $\alpha[m, n]$ ($m \geq 1$, $n \geq 2$) is also not sound and complete for any elementary class of frames. The simplest such axiom is $\alpha[1, 2]$, equivalent to $\mathsf{E}((\Box q_0 \to q_0) \wedge (\Box q_1 \to q_1))$, and expressing that its frames have chromatic number at least 3. However, algebraic results [32, theorem 4.2] can be used to show that no such logic is canonical.

Lemma 3.8. EG *has the finite model property and, for a suitable choice of the G_n, is decidable.*

Proof. Let φ be an EG-consistent formula; we will show that φ is satisfiable in a finite frame for EG.

The consistency of φ implies that φ is satisfiable in some point Γ of the canonical frame K. Let C be the cluster of M to which Γ belongs; in the proof of lemma 3.5 we already saw that C (seen as a subframe of K) is a frame for EG. Hence we are done in the case that C is finite.

If C is infinite then it contains an $R_\Box$-reflexive point. Now let M_C be the canonical model restricted to C, and take any filtration M_C^f of M_C through the collection of subformulas of φ (as in [2, §2.3]). It is a routine exercise to verify that φ is satisfiable in M_C^f, and that M_C^f is based on a standard, finite frame containing a reflexive world. But any such frame validates EG.

The proof of the second part of the lemma is done in the usual way, by choosing the G_n so that the axioms of EG are recursively enumerable, and observing that it is then decidable whether a finite frame validates the axioms. See corollary 2.16 and [2, theorem 6.7] for similar arguments. $\Box$

Remark 3.9. Fine formulated his theorem concerning the canonicity of elementarily determined modal logics in a monomodal language, i.e., with a single diamond. However, as he mentions in the introduction to [10], his results can be readily extended to polymodal logics, such as tense logics.

Similarly, we have formulated our results for bimodal languages, but it is not hard to transform them to the monomodal setting, using Thomason's simulation method. Thomason [48] showed how normal, polymodal logics can be uniformly simulated by normal, monomodal ones, in a way that preserves negative properties such as incompleteness. A systematic study of the Thomason simulation by Kracht and Wolter [40] brought out that in fact it preserves many properties, both positive and negative. Using their results, it almost immediately follows that the monomodal simulation of the logic EG is a canonical, but not elementarily determined, modal logic in a monomodal language.

In the companion paper [24], we will discuss examples of monomodal logics above K4, obtained by a direct construction not using the Thomason simulation, which are canonical but not sound and complete for any elementary class of Kripke frames.

§4. Further work. It would be interesting to know whether theorem 2.18 and the results of §3 remain true under stronger conditions. In this regard, we point out an observation and two problems. We state them in algebraic terms, but of course the modal approach could be used instead.

PROPOSITION 4.1. *The following are equivalent:*

1. *Every finitely axiomatisable canonical variety of BAOs is elementarily generated.*
2. *Every variety of BAOs with a canonical equational axiomatisation is elementarily generated.*

Proof. It is clear that (2) $\Rightarrow$ (1), since if $\mathcal{V}$ is canonical and axiomatised by finitely many equations $t_1 = u_1, \ldots, t_n = u_n$, then it is in fact axiomatisable by a single equation $(t_1 - u_1) + (u_1 - t_1) + \cdots + (u_n - t_n) = 0$, which must therefore be canonical.

Conversely, assume (1). Let $\mathcal{V}$ be a variety of L-BAOs (for some signature L) axiomatised by a set Σ of canonical equations. (Of course, $\mathcal{V}$ is canonical.) For $\varepsilon \in \Sigma$, let $\mathcal{V}_\varepsilon$ be the variety of all L-BAOs satisfying ε. If L_ε is a finite subsignature of L containing the symbols of ε, then the class $\mathcal{V}'_\varepsilon$ of L_ε-reducts of BAOs in $\mathcal{V}_\varepsilon$ is a finitely axiomatisable canonical variety, and hence by assumption is elementarily generated. By proposition 2.17, there is an elementary class $\mathcal{K}'_\varepsilon$ of L^a_ε-structures satisfying $\operatorname{Cst} \mathcal{V}'_\varepsilon \subseteq \mathcal{K}'_\varepsilon \subseteq \operatorname{Str} \mathcal{V}'_\varepsilon$. Let $\mathcal{K}_\varepsilon$ be the class of L^a-structures with L^a_ε-reducts in $\mathcal{K}'_\varepsilon$. It is easily checked that $\operatorname{Cst} \mathcal{V}_\varepsilon \subseteq \mathcal{K}_\varepsilon \subseteq \operatorname{Str} \mathcal{V}_\varepsilon$. Let $\mathcal{K} = \bigcap_{\varepsilon \in \Sigma} \mathcal{K}_\varepsilon$. Certainly, $\mathcal{K}$ is elementary. Moreover, we have

$$\operatorname{Cst} \mathcal{V} \subseteq \bigcap_{\varepsilon \in \Sigma} \operatorname{Cst} \mathcal{V}_\varepsilon \subseteq \bigcap_{\varepsilon \in \Sigma} \mathcal{K}_\varepsilon = \mathcal{K} \subseteq \bigcap_{\varepsilon \in \Sigma} \operatorname{Str} \mathcal{V}_\varepsilon = \operatorname{Str} \bigcap_{\varepsilon \in \Sigma} \mathcal{V}_\varepsilon = \operatorname{Str} \mathcal{V}.$$

By proposition 2.17, $\mathcal{V}$ is generated by $\mathcal{K}$. □

PROBLEM 4.2. *Is every finitely axiomatisable canonical variety of BAOs elementarily generated?*

PROBLEM 4.3. *Is there a variety of BAOs that is (a) canonical, (b) axiomatisable by a set of equations of the form $\Sigma \cup \Xi$, where Σ is finite and every equation in Ξ is canonical, and (c) not elementarily generated?*

The $\mathcal{V}$ of theorem 2.18 and the $\mathcal{V}_X$ of theorem 2.19 are not finitely axiomatisable. Indeed, by results in [32], any axiomatisation of them must involve infinitely many non-canonical equations.

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