# THE PRESERVATION OF SAHLQVIST EQUATIONS IN COMPLETIONS OF BOOLEAN ALGEBRAS WITH OPERATORS

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Abstract. Monk [1970] extended the notion of the completion of a Boolean algebra to Boolean algebras with operators. Under the assumption that the operators of such an algebra $\mathcal{A}$ are completely additive, he showed that the completion of $\mathcal{A}$ always exists and is unique up to isomorphisms over $\mathcal{A}$. Moreover, strictly positive equations are *preserved* under completions: a strictly positive equation that holds in $\mathcal{A}$ must hold in the completion of $\mathcal{A}$.

In this paper we extend Monk's preservation theorem by proving that certain kinds of Sahlqvist equations (as well as some other types of equations and implications) are preserved under completions. An example is given which shows that arbitrary Sahlqvist equations need not be preserved.

In the study of Boolean algebras, it is often useful to pass from a given Boolean algebra $\mathcal{A}$ to an extension of $\mathcal{A}$ that is complete (in the sense that every set of elements, whether finite or infinite, has a supremum — a least upper bound — and an infimum — a greatest lower bound — in $\mathcal{A}$). Two rather different complete extensions of $\mathcal{A}$ are known from the literature. The first is the *canonical* (or *perfect*) *extension*: the Boolean algebra $\mathcal{B}$ of subsets of the collection of all ultrafilters of $\mathcal{A}$. The second is the (*MacNeille* or *Dedekind*) *completion*: the minimal Boolean algebra $\mathcal{C}$ that extends $\mathcal{A}$ and is complete. Each of these has its advantages and disadvantages. The advantage of the canonical extension $\mathcal{B}$ is that it is atomic. The disadvantage is that all proper infinite joins which do exist in $\mathcal{A}$ are "broken" in $\mathcal{B}$; more precisely, if $a$ is the supremum in $\mathcal{A}$ of an infinite set $X$, but not of any finite subset of $X$, then $a$ cannot be the supremum of $X$ in $\mathcal{B}$. (This property is sometimes called *compactness*.) The advantage of the completion $\mathcal{C}$ is that all joins which do exist in $\mathcal{A}$ are preserved in $\mathcal{C}$: if $a$ is the supremum in $\mathcal{A}$ of any (finite or infinite) set $X$, then $a$ is the supremum of $X$ in $\mathcal{C}$. The disadvantage is that $\mathcal{C}$ cannot be atomic unless $\mathcal{A}$ itself is atomic; in fact, the only atoms in $\mathcal{C}$ are the atoms of $\mathcal{A}$.

Jónsson and Tarski [1951] extended the theory of canonical extensions to Boolean algebras with additional operations that are additive in each coordinate, so-called *Boolean algebras with operators.* They showed that every Boolean algebra with operators $\mathcal{A}$ has a canonical extension $\mathcal{B}$ (an expansion of the canonical extension of the Boolean part of $\mathcal{A}$) that is unique up to isomorphisms over $\mathcal{A}$. Moreover, they also proved that every strictly positive equation — that is, every equation in which the complementation symbol does not occur — which holds in $\mathcal{A}$ must hold in $\mathcal{B}$; in technical language, strictly positive equations are *preserved* under the passage to the canonical extension. (They also noted that some implications are similarly preserved.) Sahlqvist [1975] extended the Jónsson-Tarski preservation theorem to

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a class of equations known today as *Sahlqvist* equations (see Section 3). Recently, Jónsson [1994] has given a simple and elegant treatment of the preservation theorems for canonical extensions.

The task of extending the theory of completions from Boolean algebras to Boolean algebras with operators was taken up by Monk [1970]. He showed that every Boolean algebra with operators $\mathcal{A}$ in which the operators are completely additive in each coordinate (that is, additive with respect to infinite joins — if they exist — as well as finite, non-empty joins) has a completion $\mathcal{B}$ (an expansion of the completion of the Boolean part of $\mathcal{A}$) that is unique up to isomorphisms over $\mathcal{A}$. He also proved that strictly positive equations are preserved under the passage to the completion.

The purpose of this paper is to do for completions what Sahlqvist and Jónsson did for canonical extensions, to the extent that this is possible. In particular, we shall give a treatment, along the lines of Jónsson [1994], of Monk's preservation theorems, and we shall show in Corollary 34 that certain kinds of Sahlqvist equations are preserved under that passage from a Boolean algebra with completely additive operators to its completion. This corollary generalizes a theorem in Venema [1997] which states a similar result under the additional proviso that the algebra in question be atomic. In the last section we present, in algebraic form, an example from Venema [1993] which shows that arbitrary Sahlqvist equations are, in general, not preserved.

Our development parallels that of Jónsson [1994]; we also need some lemmas from Monk [1970] concerning complete extensions of operators. Jónsson's approach can actually be given an axiomatic formulation so that it applies not only to canonical extensions and completions, but possibly also to other kinds of extensions that have not yet been considered.

## 1. Positive and negative terms

Fix a class $\mathsf{K}$ of algebraic structures

$$\mathcal{A} = \langle A\,,\ \leq\ ,\ -\ , f_\xi\,\rangle_{\xi<\alpha}$$

of a given similarity type, where $\leq$ is a binary relation (on $A$), $-$ is a unary operation, and each $f_\xi$ is an operation of rank $n_\xi \geq 0$. (The relation $\leq$ does not play a role in this section, but it will in Section 3.) Operations of rank 0 will be identified with *distinguished elements* of $A$. We shall refer to $-$ as *negation* although it may have very few of the properties that are commonly associated with negation or complementation. The *law of double negation*,

$$- - x = x \qquad \text{for all } x \text{ in } A\,,$$

will be used frequently.

The letter $T$ shall be used to denote the set of all terms in some language appropriate for the structures of $\mathsf{K}$. Terms in which no variables occur are called *constant* (or *nullary*) *terms*. A term in which exactly one variable occurs (perhaps many times) is called a *unary* term.

The term that is the concatenation of the operation symbol $f_\xi$ and the variables $v_0, \ldots, v_{n_\xi-1}$, in that order, will be denoted by

$$f_\xi(v_0, \ldots, v_{n_\xi-1})\,.$$

For an arbitrary term $\tau$ in $T$ the notation

$$\tau(v_0, \ldots, v_{m-1})$$

is meant to indicate that the distinct variables occurring in $\tau$ are among the variables $v_0, \ldots, v_{m-1}$. If $\sigma_0, \ldots, \sigma_{m-1}$ are also terms in $T$, then the notation

$$\tau(\sigma_0, \ldots, \sigma_{m-1})$$

denotes the term obtained by simultaneously substituting $\sigma_0, \ldots, \sigma_{m-1}$ for the variables $v_0, \ldots, v_{m-1}$ in $\tau$. A subset $S$ of $T$ is said to be *closed under substitution* just in case, whenever $\tau(v_0, \ldots, v_{m-1})$ and $\sigma_0, \ldots, \sigma_{m-1}$ are terms in $S$, then $\tau(\sigma_0, \ldots, \sigma_{m-1})$ is also in $S$.

There are various notions of "positive" and "negative" that can be applied to terms.

**Definition 1.** (i) A term in $T$ is *positive primitive* if it is either a variable, a constant term, or else has the form

$$f_\xi(v_0, \ldots, v_{n_\xi - 1})$$

for some $\xi < \alpha$.

(ii) A term in $T$ is *strictly positive* if it is either a variable, a constant term, or else has the form

$$f_\xi(\sigma_0, \ldots, \sigma_{n_\xi - 1})$$

for some $\xi < \alpha$ and some strictly positive terms $\sigma_0, \ldots, \sigma_{n_\xi - 1}$.

(iii) A term in $T$ is *positive* (respectively, *negative*) if it is a variable or a constant term (respectively, a constant term) or else has one of the following two forms:

$$f_\xi(\sigma_0, \ldots, \sigma_{n_\xi - 1}),$$

where $\sigma_0, \ldots, \sigma_{n_\xi - 1}$ are all positive (respectively, negative) terms and $\xi < \alpha$;

$$-\tau,$$

where $\tau$ is a negative (respectively, positive) term.

It is obvious that every positive primitive term is strictly positive and that every strictly positive term is positive, and it is simple to show that none of the reverse implications hold. For example, the term $-f_\xi(-v_0, \ldots, -v_{n_\xi - 1})$ is positive, but not strictly positive.

Here is an equivalent formulation of the notion of a strictly positive term: no subterm beginning with the negation symbol contains an occurrence of a variable. In other words, the negation symbol does not occur in the term unless it is part of a constant term. In Henkin-Monk-Tarski [1971], p. 440, such terms are called *positive in the wider sense*.

Both de Rijke-Venema [1995] and Jónsson [1994] adopt the definition that a term is positive if each variable is in the "scope" of an even number of negations. We have defined the notions of positive and negative by an interdependent recursion. The two definitions are readily shown to be equivalent.

**Lemma 1.** *Suppose that $\rho(v_0, \ldots, v_{m-1})$ and $\gamma_0, \ldots, \gamma_{m-1}$ are terms in $T$. Then the term*

$$\rho(\gamma_0, \ldots, \gamma_{m-1})$$

*is positive if $\rho$ and $\gamma_0, \ldots, \gamma_{m-1}$ are all positive or all negative, and it is negative if either $\rho$ is positive and $\gamma_0, \ldots, \gamma_{m-1}$ are negative or else $\rho$ is negative and $\gamma_0, \ldots, \gamma_{m-1}$ are positive.*

*Proof.* The proof is by induction on the definition of positive and negative terms, applied to $\rho$. Let $\kappa$ denote the term $\rho(\gamma_0, \ldots, \gamma_{m-1})$. If $\rho$ is a constant term, then $\kappa$ coincides with $\rho$ and is both positive and negative (by definition), so the lemma holds. If $\rho$ is a variable, say $v_i$ , then $\kappa$ coincides with $\gamma_i$ , and hence it is positive or negative according as $\gamma_i$ is positive or negative.

Suppose that $\rho$ has the form

$$f_\xi(\sigma_0, \ldots, \sigma_{n_\xi - 1}) .$$

Assume first that $\sigma_0, \ldots, \sigma_{n_\xi-1}$ are all positive (so that $\rho$ is positive). If $\gamma_0, \ldots, \gamma_{m-1}$ are all positive (negative), then each term $\sigma_i(\gamma_0, \ldots, \gamma_{m-1})$ is positive (negative), by the induction hypothesis. Therefore the term

$$f_\xi(\sigma_0(\gamma_0, \ldots, \gamma_{m-1}), \ldots, \sigma_{n_\xi-1}(\gamma_0, \ldots, \gamma_{m-1}))$$

is positive (negative), by the definition of a positive (negative) term; this term is just $\kappa$. The case when $\sigma_0, \ldots, \sigma_{n_\xi-1}$ are all negative is treated similarly.

Finally, suppose that $\rho$ has the form $-\tau$. Assume first that $\tau$ is negative (so that $\rho$ is positive). If the terms $\gamma_0, \ldots, \gamma_{m-1}$ are all positive (negative), then the term $\tau(\gamma_0, \ldots, \gamma_{m-1})$ is negative (positive) by the induction hypothesis; hence, the negative of this term — that is, $\kappa$ — is positive (negative). The proof for the case when $\tau$ is positive is similar. □

It is obvious that the set of strictly positive terms is closed under substitution. One consequence of the preceding lemma is that the set of positive terms is closed under substitution.

In order to characterize the sets of positive and negative terms, we introduce the notion of the dual of a term.

**Definition 2.** The *dual* of a term $\tau(v_0, \ldots, v_{m-1})$, which is denoted by $\tau^d$, is defined to be the term $-\tau(-v_0, \ldots, -v_{m-1})$.

Notice that the dual of a constant term $\tau$ is simply $-\tau$. The dual of a variable $v_i$ is, by definition $--v_i$ ; under the law of double negation, this is of course equivalent to $v_i$. Here is a more interesting example of the dual of a term: in Boolean algebra the dual of the term $v_0 + v_1$ is the term $-(-v_0 + -v_1)$, which is often taken as the definition of $v_0 \cdot v_1$.

**Lemma 2.** *If a term is positive* (*negative*), *then its dual is positive* (*negative*).

*Proof.* The proof is by induction on the definition of positive and negative terms. Let $\rho(v_0, \ldots, v_{m-1})$ be a term in $T$. If $\rho$ is a constant term, then it and its dual are positive and also negative (by definition), so the conclusion of the lemma is trivial. If $\rho$ is a variable, say $v_i$ , then $\rho^d$ is $- - v_i$ . Since $-v_i$ is negative, by definition, the term $\rho^d$ is positive, by definition.

Now suppose that $\rho$ has the form

$$f_\xi(\sigma_0, \ldots, \sigma_{n_\xi - 1}) .$$

If each term $\sigma_i$ is positive (negative), then $\sigma_i(-v_0, \ldots, -v_{m-1})$ is negative (positive) by Lemma 1. Therefore,

$$f_\xi(\sigma_0(-v_0, \ldots, -v_{m-1}), \ldots, \sigma_{n_\xi-1}(-v_0, \ldots, -v_{m-1}))$$

is negative (positive), by Definition 1. Since $\rho^d$ is just the negation of this last term, it will be positive (negative).

Finally, suppose that $\rho$ has the form $-\tau$. If $\tau$ is negative (positive), that is, if $\rho$ is positive (negative), then $\tau(-v_0, \ldots, -v_{m-1})$ is positive (negative) by Lemma 1. Since $\rho^d$ is the negation of the negation of this last term, it will also be positive (negative), by Definition 1. □

**Corollary 3.** *Under the law of double negation, a term is equivalent to a positive (negative) term if and only if its dual is equivalent to a positive (negative) term.*

*Proof.* If a term $\tau$ is positive (negative), then $\tau^d$ is positive (negative) by the previous lemma. If $\tau^d$ is positive (negative) then $(\tau^d)^d$ is positive (negative). But $(\tau^d)^d$ is equivalent to $\tau$ under the law of double negation. Therefore $\tau$ is equivalent to a positive (negative) term. □

A proof of the preceeding corollary can also be based on the following simple observation (the proof of which we leave to the reader). It asserts that the substitution of duals of terms into the dual of a term is equivalent to the dual of a term.

**Lemma 4.** *For all terms $\tau(v_0, \ldots, v_{m-1})$ and $\sigma_0, \ldots, \sigma_{m-1}$ in $T$ the equality*

$$\tau^d({\sigma_0}^d, \ldots, {\sigma_{n-1}}^d) \approx [\tau(\sigma_0, \ldots, \sigma_{n-1})]^d$$

*is derivable from the law of double negation.*

The next remark and the subsequent lemma are intended to clarify what is meant by the dual of a term. For a moment, assume that our language has been expanded to include the symbols $f_\xi^d$ (for $\xi < \alpha$) as basic operation symbols. Take $\Xi$ to be the set of axioms consisting of the equations

$$f_\xi^d(v_0, \ldots, v_{n_\xi - 1}) = -f_\xi(-v_0, \ldots, -v_{n_\xi - 1})$$

and the law of double negation. We shall say that a term is in *standard form* if the only negation symbols that occur in it occur as parts of constant terms or occur next to variables (that is, negation symbols only have constant terms or variables as arguments). Then every term $\tau$ is equivalent (on the basis of $\Xi$) to a term in standard form. The proof is by induction on terms. The key idea is that we can use the equations in $\Xi$ to bring all negations through to the variables, and then use the law of double negation to cancel as many of the negations as possible. For example, the terms

$$-f_\xi(\rho_0, \ldots, \rho_{n_\xi - 1}) \qquad \text{and} \qquad f_\xi^d(-\rho_0, \ldots, -\rho_{n_\xi - 1})$$

are equivalent (on the basis of $\Xi$), so in the left-hand term we can bring the negation symbol inside to the terms $\rho_i$ if we change $f_\xi$ to $f_\xi^d$. Similarly, the terms

$$-f_\xi^d(\rho_0, \ldots, \rho_{n_\xi - 1}) \qquad \text{and} \qquad f_\xi(-\rho_0, \ldots, -\rho_{n_\xi - 1})$$

are equivalent, so in the left-hand term we can bring the negation symbol inside to the terms $\rho_i$ if we change $f_\xi^d$ to $f_\xi$. We leave the details to the reader.

**Lemma 5.** *Suppose that $\tau$ is a term in the expanded language and that $\tau^*$ is the term obtained from $\tau$ by interchanging all occurrences of $f_\xi$ and $f_\xi^d$ for each $\xi < \alpha$. Then $\tau^d$ is equivalent to $\tau^*$ on the basis of $\Xi$.*

*Proof.* Again, the proof is by induction on terms. Here is the main idea. Since $\tau^d$ is the term $-\tau(-v_0, \ldots, -v_{m-1})$, we can bring the first negation symbol all the way through, just as was described above; in this process, each occurrence of $f_\xi$

becomes an occurrence of $f_\xi^d$, and conversely. At the end of the process, we end up with the term

$$\tau^*(- - v_0, \ldots, - - v_{m-1}),$$

which of course is equivalent to $\tau^*$. The details are left to the reader. □

Of course the symbols $f_\xi^d$ need not be basic operation symbols of our language. The preceding lemma is still true (on the basis of the law of double negation alone) provided that we understand $\tau^*$ in the proper way. Namely, we always interpret a term of the form

$$-f_\xi(-\rho_0, \ldots, -\rho_{n_\xi - 1})$$

as

$$f_\xi^d(\rho_0, \ldots, \rho_{n_\xi - 1}).$$

Here is a characterization of the strictly positive terms: a term is strictly positive if and only if it belongs to every set of terms that contains the positive primitive terms and is closed under substitution. The proof is a simple induction on the definition of a strictly positive term. A similar characterization holds for positive terms. To formulate it, let $\Psi$ be the smallest set of terms that contains all positive primitive terms and their duals, and is closed under substitution. Then a term is equivalent to a positive term (under the law of double negation) if and only if it is equivalent to a term in $\Psi$. The proof of this characterization is somewhat more involved than in the case of strictly positive terms. It is actually necessary to formulate a dual statement, just as was done in Lemma 1. The key implication, namely the one from left to right in part (i), is Lemma 5.4 in Jónsson [1994].

**Theorem 6.** *Assume that the law of double negation holds.*

(i) *A term is equivalent to a positive term if and only if it is equivalent to a term in $\Psi$.*
(ii) *A term is equivalent to a negative term if and only if it is equivalent to the negation of a term in $\Psi$.*

*Proof.* The idea of the proof is similar to the proofs sketched previously. Because this theorem plays an important role in our further development, we give a careful proof (by induction on terms). Let $\Gamma$ be the set of terms that are equivalent (under the law of double negation) to a term in $\Psi$, and $\Delta$ the set of terms that are equivalent to the negation of a term in $\Psi$. We shall show that $\Gamma$ coincides with the set of terms equivalent to some positive term, and $\Delta$ coincides with the set of terms equivalent to some negative term.

Certainly the set of positive terms contains the positive primitive terms and their duals (by Lemma 2) and is closed under substitution (by Lemma 1). Therefore $\Gamma$ is included in the set of terms equivalent to some positive term, and $\Delta$ is included in the set of terms equivalent to some negative term.

To show the reverse inclusions we proceed by induction on the definition of positive and negative terms. Certainly all constant terms and variables are in $\Gamma$ since they are positive primitive. Also, every constant term is in $\Delta$ since it equivalent to the negation of a term in $\Psi$ (it is equivalent to the negation of its own negation).

Suppose that a term $\rho$ has the form

$$f_\xi(\sigma_0, \ldots, \sigma_{n_\xi - 1}).$$

Consider first the case when each term $\sigma_i$ is positive. By the induction hypothesis $\sigma_i$ is in $\Gamma$ and hence is equivalent to a term $\gamma_i$ of $\Psi$. Then $\rho$ is equivalent to the term

$$f_\xi(\gamma_0, \ldots, \gamma_{n_\xi - 1}) ,$$

which is obviously in $\Psi$ (it is a substitution instance of the positive primitive term $f_\xi(v_0, \ldots, v_{n_\xi-1})$ and terms in $\Psi$). If $\sigma_i$ is negative (for each $i$), then by the induction hypothesis it is in $\Delta$, and hence is equivalent to a term $-\gamma_i$ for some term $\gamma_i$ in $\Psi$. Then $\rho$ is equivalent to the term

$$f_\xi(-\gamma_0, \ldots, -\gamma_{n_\xi - 1}) ,$$

which is obviously equivalent to

$$- - f_\xi(-\gamma_0, \ldots, -\gamma_{n_\xi - 1}) ,$$

that is, to

$$-f_\xi^d(\gamma_0, \ldots, \gamma_{n_\xi - 1}) .$$

Since

$$f_\xi^d(\gamma_0, \ldots, \gamma_{n_\xi - 1})$$

is in $\Psi$, the term $\rho$ is equivalent to the negation of a term in $\Psi$, and hence is in $\Delta$.

Finally, consider the case when $\rho$ has the form $-\tau$. If $\tau$ is negative, then it is in $\Delta$ (by the induction hypothesis) and hence equivalent to a term of the form $-\gamma$ for some term $\gamma$ in $\Psi$. Therefore $\rho$ is equivalent to $\gamma$, and hence is in $\Gamma$. If $\tau$ is positive, then it is in $\Gamma$ and hence equivalent to a term $\gamma$ in $\Psi$. Obviously, $\rho$ is equivalent to $-\gamma$, and hence is in $\Delta$. □

**Corollary 7.** *A term $\rho(v_0, \ldots, v_{m-1})$ is negative if and only if it is equivalent to a term of the form $\gamma(-v_0, \ldots, -v_{m-1})$ for some $\gamma$ in $\Psi$.*

*Proof.* A term $\rho$ is negative if and only if it is equivalent to $-\tau$ for some term $\tau$ in $\Psi$, by the preceding theorem. Since $\tau$ is in $\Psi$, it is in standard form. Of course $\tau^*$ (constructed in the proof of Lemma 5) is also in $\Psi$. The term $-\tau(-v_0, \ldots, -v_{m-1})$ is equivalent to $\tau^*$ by Lemma 5. Therefore, $-\tau(v_0, \ldots, v_{m-1})$ is equivalent to $\tau^*(-v_0, \ldots, -v_{m-1})$. □

Here is another way of thinking about Theorem 6(i): it says that a term is positive if and only if it is equivalent (on the basis of $\Xi$) to a strictly positive term in the language that contains the symbols $f_\xi^d$ as basic operation symbols.

For each term $\tau(v_0, \ldots, v_{m-1})$ of $T$ and each structure $\mathcal{A}$ in $\mathsf{K}$, let $\tau^{\mathcal{A}}$ denote the operation on $A$ of rank $m$ induced by $\tau$. For each operation $g$ on $A$ of rank $m$ let $g^d$ denote the algebraic dual of the operation $g$, that is, the operation of rank $m$ on $A$ determined by the rule

$$g^d(x_0, \ldots, x_{m-1}) = -g(-x_0, \ldots, -x_{m-1})$$

for all $x_0, \ldots, x_{m-1}$ in $A$.

**Lemma 8.** *Under the preceding hypotheses we have*

$$(\tau^{\mathcal{A}})^d = (\tau^d)^{\mathcal{A}} .$$

In other words, the dual of the operation induced by $\tau$ in $\mathcal{A}$ is the same as the operation induced in $\mathcal{A}$ by the dual of the term $\tau$. This is obvious from the definitions of the dual of a term and the dual of an operation.

## 2. Completions

The completion of a Boolean algebra $\mathcal{A}$ is the minimal extension of $\mathcal{A}$ that is complete (in the sense that all sums and products of infinite sets of elements exist). Mangani [1966] introduced the notion of the completion of a cylindric algebra and showed that it is again a cylindric algebra. Monk [1970] carried out a general study of completions of Boolean algebras with completely additive operators and proved that strictly positive equations are preserved under completions. To extend this theorem to other kinds of equations, we shall need a number of properties of so-called "complete extensions" of operators on a Boolean algebra. Most of these properties are proved in Monk's paper or else in Jónsson-Tarski [1951]. Indeed, Lemmas 9 and 19 below occur as Theorems 1.6 and 1.14 in Jónsson-Tarski [1951]; Lemmas 11, 13–16, 20 occur as Theorems 1.1, 1.2, 1.5, 1.8, and Corollary 1.11 in Monk [1970] (and these are, in turn, the analogues, for completions, of theorems proved for canonical extensions in Jónsson-Tarski [1951]). To save the reader the time and trouble of consulting these other papers, we shall illustrate the proofs with the case when the operator has rank 2.

Lemmas 18 and 21 are the essential new results in this section; they are the analogues for dual operations of Lemmas 15 and 16.

Let $\mathcal{A}$ be a Boolean algebra and $A$ its universe. An operation $h$ of rank $n$ on $A$ is said to be *completely additive* if it is completely additive in each coordinate. For instance, a binary operation $h$ on $A$ is completely additive if, for all elements $y$ in $A$ and all non-empty subsets $X$ of $A$ such that $\sum X$ exists, the sums

$$\sum\{h(x,y) : x \in X\} \qquad , \qquad \sum\{h(y,x) : x \in X\}$$

exist, and

$$h(\sum X, y) = \sum\{h(x,y) : x \in X\} \qquad , \qquad h(y, \sum X) = \sum\{h(y,x) : x \in X\}\,.$$

**Lemma 9.** *If $h$ is a completely additive operation of rank $n$ on $A$ and if, for each $i < n$, the set $X_i$ is a non-empty subset of $A$ such that $\sum X_i$ exists, then*

$$h(\sum X_0, \ldots, \sum X_{n-1}) = \sum\{h(x_0, \ldots, x_{n-1}) : x_i \in X_i \textit{ for } i < n\}\,.$$

*Proof.* We illustrate the proof for the case $n = 2$. By invoking the complete additivity of $h$, first for the zeroth coordinate and then for the first coordinate, we obtain

$$\begin{aligned} h(\sum X_0, \sum X_1) &= \sum\{h(x_0, \sum X_1) : x_0 \in X_0\} \\ &= \sum\{\sum\{h(x_0, x_1) : x_1 \in X_1\} : x_0 \in X_0\} \\ &= \sum\{h(x_0, x_1) : x_0 \in X_0 \text{ and } x_1 \in X_1\}\,. \end{aligned}$$

□

The *completion* of the Boolean algebra $\mathcal{A}$ is a complete Boolean algebra $\mathcal{A}^+$ such that $\mathcal{A}$ is a subalgebra of $\mathcal{A}^+$ and each element of $\mathcal{A}^+$ is the sum (in $\mathcal{A}^+$) of the elements in $\mathcal{A}$ that are below it, in symbols

$$x = \sum\{a \in A : a \le x\}.$$

The completion of $\mathcal{A}$ always exists and is unique up to isomorphisms over $\mathcal{A}$ (see Section 35 in Sikorski [1964]). Moreover, if $X$ is a subset of $A$ whose sum (supremum) exists in $\mathcal{A}$, then this sum coincides with the sum of $X$ in $\mathcal{A}^+$. A proof that the completion of $\mathcal{A}$ is just the minimal complete extension of $\mathcal{A}$ is given, for example, in Theorem 35.2 of *op. cit.*

We continue to fix a Boolean algebra $\mathcal{A}$ and its completion $\mathcal{A}^+$.

**Lemma 10.** *If $h$ and $k$ are completely additive operations on $A^+$ (of the same rank) that agree on $A$, then they agree on $A^+$.*

*Proof.* For $i = 0, 1$, let $X_i$ be a non-empty subset of $A$ such that the sum $x_i = \sum X_i$ exists. Using the previous lemma and the assumption that $h$ and $k$ agree on $A$, we get

$$\begin{aligned} h(x_0, x_1) &= \sum\{h(a_0, a_1) : a_0 \in X_0 \text{ and } a_1 \in X_1\} \\ &= \sum\{k(a_0, a_1) : a_0 \in X_0 \text{ and } a_1 \in X_1\} \\ &= k(x_0, x_1)\,. \end{aligned}$$

□

The operations and relations on a Boolean algebra $\mathcal{A}$ extend in a natural way to operations and relations on powers of $\mathcal{A}$: the extensions are defined coordinatewise, of course. For example, if $x$ and $y$ are sequences in ${}^nA$ (the set of all sequences of length $n$ whose terms are in $A$), then write $x \leq y$ just in case $x_i \leq y_i$ for $i < n$, and write

$$-x = (-x_0, \ldots, -x_{n-1}) \qquad , \qquad x + y = (x_0 + y_0, \ldots, x_{n-1} + y_{n-1})\,;$$

if $f$ and $g$ are operations on $A$ of the same rank $n$, then write $f \leq g$ just in case $f(x) \leq g(x)$ for each $x$ in ${}^nA$. These extensions allow some simplification of notation.

**Definition 3.** The *complete extension* of an operation $h$ on $A$ of rank $n$ is the operation $h^+$ on $A^+$ of rank $n$ that is defined by

$$h^+(x) = \sum\{h(a) : a \in {}^nA \text{ and } a \leq x\}$$

for every sequence $x$ of length $n$ from $A^+$.

Recall that an operation $h$ on $A$ of rank $n$ is said to be *monotone* (or *isotone*) if $h(x) \leq h(y)$ for any two $x, y$ in ${}^nA$ with $x \leq y$. The proof of the next lemma is easy and is left to the reader.

**Lemma 11.** *If $h$ is a monotone operation on $A$, then $h^+$ is a monotone operation on $A^+$ and its restriction to $A$ is just $h$.*

Lemma 11 says that the complete extension of a monotone operation $h$ really is an extension of $h$ in the algebraic sense of the word; if $h$ is not monotonic, then that won't be true. For this reason we shall only speak about complete extensions of monotonic operations.

**Lemma 12.** *For each $i < n$, let $X_i$ be a non-empty subset of $A$ and $x_i = \sum X_i$ (in $\mathcal{A}^+$). If $h$ is a completely additive operation on $A$ of rank $n$, then*

$$h^+(x_0, \ldots, x_{n-1}) = \sum\{h(a_0, \ldots, a_{n-1}) : a_i \in X_i \textit{ for } i < n\}\,.$$

In other words, in computing $h^+(x_0, \ldots, x_{n-1})$ we do not have to sum over all of the elements $a_i$ that are below $x_i$ and in $A$; we only have to sum over those $a_i$ that are below $x_i$ and in $X_i$.

*Proof.* For $i = 0, 1$ set

$$A_i = \{a \in A : a \leq x_i\}\,.$$

Then $X_i \subseteq A_i$ and therefore

$$\begin{aligned} h^+(x_0, x_1) &= \textstyle\sum\{h(a_0, a_1) : a_0 \in A_0 \text{ and } a_1 \in A_1\} \\ &\geq \textstyle\sum\{h(b, c) : b \in X_0 \text{ and } c \in X_1\}\,, \end{aligned}$$

by the definition of $h^+$. To prove the reverse inequality, suppose that $a_i \in A_i$ for $i = 0, 1$. Then $a_i \leq x_i$, so

$$a_i = a_i \cdot x_i = a_i \cdot \textstyle\sum X_i = \sum\{a_i \cdot b : b \in X_i\}\,.$$

Hence

$$\begin{aligned} h(a_0, a_1) &= \textstyle\sum\{h(a_0 \cdot b, a_1 \cdot c) : b \in X_0 \text{ and } c \in X_1\} \\ &\leq \textstyle\sum\{h(b, c) : b \in X_0 \text{ and } c \in X_1\} \end{aligned}$$

by Lemma 9 and the monotonicity of $h$. Summing over all $a_i$ in $A_i$ , we get

$$\textstyle\sum\{h(a_0, a_1) : a_0 \in A_0 \text{ and } a_1 \in A_1\} \leq \sum\{h(b, c) : b \in X_0 \text{ and } c \in X_1\}\,.$$

Since the left-hand side is just $h^+(x_0, x_1)$, this is the desired inequality. □

**Lemma 13.** *If $h$ is a completely additive operation on $A$, then $h^+$ is completely additive on $A^+$.*

*Proof.* We illustrate the proof (in the case of rank 2) by showing that $h^+$ is additive in the zeroth coordinate. Fix a non-empty subset $X$ of $A^+$ and set $u = \sum X$. For each $x$ in $A^+$ put $A_x = \{a \in A : a \leq x\}$. Then $x = \sum A_x$ , since $\mathcal{A}^+$ is the completion of $\mathcal{A}$, and therefore $u = \sum(\bigcup_{x\in X} A_x)$. (However, it is not necessarily true that $A_u = \bigcup_{x\in X} A_x$.) Thus, for any element $v$ of $A^+$ we have:

$$\begin{aligned} h^+(u, v) &= \textstyle\sum\{h(a, b) : a \in \bigcup_{x\in X} A_x \text{ and } b \in A_v\} \\ &= \textstyle\sum\{\sum\{h(a, b) : a \in A_x \text{ and } b \in A_v\} : x \in X\} \\ &= \textstyle\sum\{h^+(x, v) : x \in X\}\,. \end{aligned}$$

The first equality follows by Lemma 12 (with $n = 2$ and the sets $\bigcup_{x\in X} A_x$ and $A_v$ in place of $X_0$ and $X_1$), the second by Boolean algebra, and the third by the definition of $h^+$. □

An operation $h$ on $A^+$ (or $A$) of rank $n$ is *constant* if $h(x) = h(y)$ for all sequences $x, y$ of length $n$; $h$ is the $i^{\text{th}}$ *projection function* if $h(x) = x_i$ for each sequence $x = (x_0, \ldots, x_{n-1})$. The unary operation $h$ satisfying $h(0) = 0$ and $h(x) = 1$ for $x \neq 0$ is called the *unary discriminator.* All such operations, as well as Boolean addition and multiplication, are well known to be completely additive. It follows that these operations on $A^+$ coincide with the complete extensions of the corresponding operations on $A$. Indeed, the complete extension of a completely additive operation on $A$ is completely additive, by the preceding lemma, and two completely additive operations that agree on $A$ must agree on $A^+$, by Lemma 10. Summarizing:

**Lemma 14.** *Each of the following operations on $A^+$ coincides with the complete extension of the corresponding operation on $A$: Boolean addition, Boolean multiplication, the unary discriminator, any constant operation, and any projection operation.*

If $h$ is an arbitrary operation of rank $n$ on the set $A$ and if $g_0, \ldots, g_{n-1}$ are operations of the same rank $m$ on $A$, then the *composition* of $h$ with $g_0, \ldots, g_{n-1}$ is defined to be the operation

$$h[g_0, \ldots, g_{n-1}]$$

of rank $m$ on $A$ determined by

$$h[g_0, \ldots, g_{n-1}](x) = h(g_0(x), \ldots, g_{n-1}(x))$$

for each $x$ in ${}^mA$.

**Lemma 15.** *If $h$ is a monotone operation on $A$ of rank $n$ and if $g_0, \ldots, g_{n-1}$ are monotone operations on $A$ of rank $m$, then*

$$h[g_0, \ldots, g_{n-1}]^+ \leq h^+[g_0^+, \ldots, g_{n-1}^+]\,.$$

*Proof.* For each sequence $x$ of length $m$ from $A^+$ we have (in the case $n = 2$)

$$\begin{aligned} h[g_0, g_1]^+(x) &= \textstyle\sum\{h[g_0, g_1](a) : a \in {}^mA \text{ and } a \leq x\} \\ &= \textstyle\sum\{h(g_0(a), g_1(a)) : a \in {}^mA \text{ and } a \leq x\} \\ &= \textstyle\sum\{h^+(g_0^+(a), g_1^+(a)) : a \in {}^mA \text{ and } a \leq x\} \\ &\leq h^+(g_0^+(x), g_1^+(x)) \\ &= h^+[g_0^+, g_1^+](x)\,. \end{aligned}$$

The first step follows from the definition of the complete extension of $h[g_0, g_1]$, the second and last from the definition of functional composition, and the third and fourth from Lemma 11. □

The reverse inequality holds in the special case when $h$ is completely additive.

**Lemma 16.** *If $h$ is a completely additive operation on $A$ of rank $n$ and if $g_0, \ldots, g_{n-1}$ are monotone operations on $A$ of rank $m$, then*

$$h^+[g_0^+, \ldots, g_{n-1}^+] \leq h[g_0, \ldots, g_{n-1}]^+\,.$$

*Proof.* Fix a sequence $x$ of length $m$ from $A^+$ and set

$$S_x = \{a \in {}^mA : a \leq x\}\,.$$

In the case $n = 2$ we have

$$\begin{aligned} h^+[g_0^+, g_1^+](x) &= h^+(g_0^+(x), g_1^+(x)) \\ &= h^+(\textstyle\sum\{g_0(a_0) : a_0 \in S_x\}, \textstyle\sum\{g_1(a_1) : a_1 \in S_x\}) \\ &= \textstyle\sum\{h^+(g_0(a_0), g_1(a_1)) : a_0, a_1 \in S_x\} \\ &= \textstyle\sum\{h(g_0(a_0), g_1(a_1)) : a_0, a_1 \in S_x\} \end{aligned}$$

The first equality follows from the definition of functional composition, the second from the definition of the complete extension $g_i^+$, the third from Lemma 9 and the complete additivity of $h^+$ (here we use the complete additivity of $h$ and Lemma 13), and the fourth from Lemma 11.

Observe that $S_x$ contains the zero sequence and is closed under finite Boolean sums. From this it follows that

$$S_x = \{a_0 + a_1 : a_0, a_1 \in S_x\}\,. \tag{1}$$

Therefore,

$$\begin{aligned}\textstyle\sum\{h(g_0(a_0), g_1(a_1)) : a_0, a_1 \in S_x\} &\leq \textstyle\sum\{h(g_0(a_0 + a_1), g_1(a_0 + a_1)) : a_0, a_1 \in S_x\} \\ &= \textstyle\sum\{h(g_0(a), g_1(a)) : a \in S_x\} \\ &= \textstyle\sum\{h[g_0, g_1](a) : a \in S_x\} \\ &= h[g_0, g_1]^+(x)\end{aligned}$$

by monotony, (1), the definition of functional composition, and the definition of the complete extension of $h[g_0, g_1]$. □

Following Jónsson [1994], we say that an operation $h$ on $A$ of rank $n$ is *conservative* if

$$h[g_0, \ldots, g_{n-1}]^+ = h^+[g_0^+, \ldots, g_{n-1}^+]$$

whenever $g_0, \ldots, g_{n-1}$ are monotone operations on $A$ of the same rank $m$. Together, Lemmas 15 and 16 assert that a completely additive operation $h$ on $A$ is conservative.

We now turn to the study of complete extensions of dual operations.

**Lemma 17.** *If $h$ is a monotone operation, then so is $h^d$.*

Lemma 17 ensures us that it is reasonable to form the complete extension of a dual operation.

*Proof.* Assume that $h$ has rank $n$, and let $x$ and $y$ be sequences in ${}^nA$ such that $x \leq y$. Then $-y \leq -x$ and therefore $h(-y) \leq h(-x)$, since $h$ is monotone. Hence, $-h(-x) \leq -h(-y)$. In other words, $h^d(x) \leq h^d(y)$. □

**Lemma 18.** *If $h$ is a monotone operation on $A$, then $(h^d)^+ \leq (h^+)^d$*

*Proof.* Assume that $h$ has rank $n$ and that $x$ is an $n$-termed sequence of elements from $A^+$ . Since

$$h^+(y) = \textstyle\sum\{h(a) : a \in {}^nA \text{ and } a \leq y\}$$

for any sequence $y$ of elements from $A^+$ (by definition of $h^+$), it follows that

$$h^+(-x) = \textstyle\sum\{h(a) : a \in {}^nA \text{ and } a \leq -x\}$$

(take $y = -x$). On the other hand,

$$(h^d)^+(x) = \textstyle\sum\{h^d(b) : b \in {}^nA \text{ and } b \leq x\}$$

by definition. Hence,

$$(1) \qquad h^+(-x) \cdot (h^d)^+(x) = \textstyle\sum\{h(a) \cdot h^d(b) : a, b \in {}^nA \text{ and } a \leq -x \ , \ b \leq x\} \, ,$$

by the complete additivity of Boolean multiplication. Suppose that $a \leq -x$ and $b \leq x$. Then $a \leq -x \leq -b$ and therefore $h(a) \leq h(-b)$, by monotony. Consequently,

$$h(a) \cdot h^d(b) = h(a) \cdot -h(-b) = 0 \, .$$

It follows that each summand on the right-hand side of (1) is 0, so

$$h^+(-x) \cdot (h^d)^+(x) = 0 \, .$$

In other words,

$$(h^d)^+(x) \leq -h^+(-x) = (h^+)^d(x) \, ,$$

as was to be shown. □

The reverse inequality in Lemma 18 does not hold in general, even when $h$ is completely additive.[1] However, it does hold under the stronger assumption that $h$ is conjugated. Recall from Jónsson-Tarski [1951] that two unary operations $h$ and $g$ on $A$ are *conjugates* in case

$$x \cdot h(y) = 0 \quad \text{if and only if} \quad g(x) \cdot y = 0$$

for all elements $x, y$ in $A$. A unary operation $h$ on $A$ is *conjugated* if it has a conjugate. As an example, consider the unary discriminator $k$ on $A$. Since

$$x \cdot k(y) = 0 \quad \text{if and only if} \quad x = 0 \text{ or } y = 0\,,$$

the operation $k$ is its own conjugate. In particular, $k$ is conjugated. (It is possible to extend the notion of conjugates to operations of higher rank. For instance, two binary operations $h$ and $g$ on a set $A$ are *left conjugates*, or *zeroth coordinate conjugates*, of one another if, for all elements $x, y, z$ in $A$, we have

$$x \cdot h(y, z) = 0 \quad \text{iff} \quad g(x, z) \cdot y = 0\,.$$

Right conjugates can be defined in a completely analogous manner. Since conjugated operations of higher rank will play no role in our subsequent development, we shall assume, without always restating it, that conjugated operations are unary.)

**Lemma 19.** *A conjugated operation is completely additive.*

*Proof.* Suppose that $h$ is a conjugated operation on $A$ — say, $g$ is its conjugate — and let $X$ be a non-empty subset of $A$ such that the supremum $x = \sum X$ exists. Then for each element $a$ in $A$ we have

$$\begin{aligned} a \cdot h(b) = 0 \text{ for every } b \in X \quad &\text{iff} \quad g(a) \cdot b = 0 \text{ for every } b \in X \\ &\text{iff} \quad g(a) \cdot (\textstyle\sum X) = 0 \\ &\text{iff} \quad g(a) \cdot x = 0 \\ &\text{iff} \quad a \cdot h(x) = 0 \end{aligned}$$

The first and last equivalence are a consequence of conjugacy, and the second follows from the complete additivity of Boolean multiplication.

If $a = -h(x)$, then certainly $a \cdot h(x) = 0$, and therefore $a \cdot h(b) = 0$ for every element $b$ in $X$ (by the previous chain of equivalences). In other words, $h(b) \le h(x)$ for every $b$ in $X$, or, put another way, $h(x)$ is an upper bound for the set

$$H = \{h(b) : b \in X\}\,.$$

Let $y$ be any other upper bound for $H$, and set $a = -y$. Obviously $a \cdot h(b) = 0$ for every $b$ in $X$. Hence $a \cdot h(x) = 0$ (by the chain of equivalences), so $h(x) \le y$. In other words, $h(x)$ is the least upper bound for $H$. □

**Lemma 20.** *If $h$ and $g$ are conjugates, then so are $h^+$ and $g^+$.*

---

[1] This follows from the example of Venema [1993] that is given in Section 4 below. There, a completely additive unary operation $f$ on a Boolean algebra $\mathcal{A}$ is constructed with the property that the equation

$$f(f^d(v)) \le f^d(f(v)) \tag{1}$$

holds in algebra $\langle \mathcal{A}\,, f\,\rangle$, but not in its completion. If the reverse inequality in Lemma 18 were true for completely additive functions, then the inequality (1) would hold in the completion $\langle \mathcal{A}^+\,, f^+\,\rangle$, by the argument in the proof of Corollary 34(ii) below.

*Proof.* Suppose that $x_0, x_1$ are elements in $A^+$, and put $A_i = \{a \in A : a \leq x_i\}$. Then

$$\begin{aligned}
x_0 \cdot h^+(x_1) = 0 \qquad &\text{iff} \qquad \textstyle\sum\{a_0 : a_0 \in A_0\} \cdot \sum\{h(a_1) : a_1 \in A_1\} = 0 \\
&\text{iff} \qquad \textstyle\sum\{a_0 \cdot h(a_1) : a_0 \in A_0 \text{ and } a_1 \in A_1\} = 0 \\
&\text{iff} \qquad a_0 \cdot h(a_1) = 0 \quad \text{for all } a_0 \in A_0 \text{ and } a_1 \in A_1 \\
&\text{iff} \qquad g(a_0) \cdot a_1 = 0 \quad \text{for all } a_0 \in A_0 \text{ and } a_1 \in A_1 \\
&\text{iff} \qquad \textstyle\sum\{g(a_0) \cdot a_1 : a_0 \in A_0 \text{ and } a_1 \in A_1\} = 0 \\
&\text{iff} \qquad \textstyle\sum\{g(a_0) : a_0 \in A_0\} \cdot \sum\{a_1 : a_1 \in A_1\} = 0 \\
&\text{iff} \qquad g^+(x_0) \cdot x_1 = 0\,.
\end{aligned}$$

The first and last equivalence hold by the definition of a completion and of a complete extension (of an operation), the second and sixth hold by the complete additivity of Boolean multiplication, the third and fifth by Boolean algebra, and the fourth because $h$ and $g$ are assumed to be conjugates. □

The preceding lemma can also be obtained as an immediate consequence of Corollary 31(ii) below.

**Lemma 21.** *If $h$ is conjugated, then* $(h^+)^d \leq (h^d)^+$.

*Proof.* Fix any element $x$ in $A^+$. Since both $(h^+)^d(x)$ and $(h^d)^+(x)$ are elements of the completion, they are both the sums of the elements of $A$ that are beneath them. To prove the lemma, then, it suffices to show that every element of $A$ below $(h^+)^d(x)$ is below $(h^d)^+(x)$.

Let $a$ be an element of $A$ below $(h^+)^d(x)$. Then $a \leq -h^+(-x)$, by definition of $(h^+)^d$. Hence, $a \cdot h^+(-x) = 0$.

Suppose that $g$ is the conjugate of $h$. By the previous lemma, $g^+$ is the conjugate of $h^+$. Therefore $g^+(a) \cdot -x = 0$. Because $g^+$ coincides with $g$ on $A$, we see that

$$g(a) \leq x\,. \tag{1}$$

The function $h$, being completely additive (by Lemma 19), must be monotone. It follows that $h^d$ is monotone, by Lemma 17, and hence that $(h^d)^+$ is monotone, by Lemma 11. Applying $(h^d)^+$ to both sides of (1), we get

$$(h^d)^+(g(a)) \leq (h^d)^+(x)\,. \tag{2}$$

On the other hand, $g(a) \cdot -g(a) = 0$, whence, by conjugacy, $a \cdot h(-g(a)) = 0$. Thus,

$$a \leq -h(-g(a)) = h^d(g(a)) = (h^d)^+(g(a))\,.$$

Combining this with (2), we arrive at the desired conclusion: $a \leq (h^d)^+(x)$. □

Lemmas 19 and 20 and their proofs can easily be extended to conjugated operations of higher rank. However, we shall give an example in Section 4 to show that Lemma 21 fails for conjugated operations of higher rank.

Suppose that

$$\mathcal{A} = \langle A\,,\, +\,,\, -\,, f_\xi\,\rangle_{\xi<\alpha}$$

is a Boolean algebra with completely additive operators. An algebra

$$\mathcal{B} = \langle B\,,\, +\,,\, -\,, g_\xi\,\rangle_{\xi<\alpha}$$

is said to be a *completion* of $\mathcal{A}$ provided that $\langle B\,,\,+\,,-\,\rangle$ is the Boolean algebraic completion of $\langle A\,,\,+\,,-\,\rangle$, each of the operators $g_\xi$ is completely additive, and

$$g_\xi(x) = \sum\{f_\xi(a) : a \in {}^{n_\xi}A \text{ and } a \le x\}$$

for each $x$ in ${}^{n_\xi}B$. It follows from the last condition that $g_\xi$ agrees with $f_\xi$ on $A$, and hence, by Lemma 10, that $g_\xi = f_\xi^+$. Since $\langle B\,,\,+\,,-\,\rangle$ is uniquely determined up to isomorphisms over $\langle A\,,+\,,-\,\rangle$, it follows that $\mathcal{B}$ is uniquely determined, up to isomorphisms over $\mathcal{A}$. We shall therefore speak of *the* completion of $\mathcal{A}$ and denote it by

$$\mathcal{A}^+ = \langle A^+\,,\,+\,,\,-\,,f_\xi^+\,\rangle_{\xi<\alpha}\,.$$

## 3. The preservation theorems

Jónsson [1994] gives a proof that Sahlqvist equations and inequalities are preserved under canonical (that is, perfect) extensions. His approach uses ideas that go back to Jónsson-Tarski [1951], but it is much more streamlined and elegant. The approach can be adapted to obtain preservation theorems for completions. In fact, Jónsson's approach is really a kind of axiomatic approach to the preservation theorems. To make this clear, suppose for the moment that $\mathsf{K}$ is a class of structures

$$\mathcal{A} = \langle A\,,\,\le\,,\,-\,,f_\xi\,\rangle_{\xi<\alpha}$$

as in Section 1, and assume that there is a construction which associates with each reduct $\langle A\,,\,\le\,,-\,\rangle$ an extension $\langle A^+\,,\,\le\,,-\,\rangle$, and with each operation $h$ of rank $n$ on $A$ an operation $h^+$ of rank $n$ on $A^+$. Set

$$\mathcal{A}^+ = \langle A^+\,,\,\le\,,\,-\,,f_\xi^+\,\rangle_{\xi<\alpha}\,.$$

If we adopt as axioms for this construction the requirement that the operations $f_\xi$ of $\mathcal{A}$ are monotone, the law of double negation, the laws for $\le$ of transitivity and anti-monotonicity (if $x \le y$, then $-y \le -x$), and the conclusions of Lemma 11, Lemma 14 for projection and constant operations, Lemma 15, Lemma 16 for the basic operations $f_\xi$ of $\mathcal{A}$ (in place of $h$), and Lemma 18, then it is possible to derive almost all of the lemmas and theorems in Sections 1, 4, and 5 of Jónsson [1994].

Virtually all of the results in this section are the analogues for completions of results proved (explicitly or implicitly) in *op. cit.* for canonical extensions, and the same proofs work in both cases. In particular, Lemma 23, Lemma 24, Corollary 26, Lemma 27, Lemma 28, Theorem 29, Theorem 30, and Theorem 32 below are the analogues of Lemma 4.3(i),(ii), Theorem 4.4, Theorem 5.1, Theorem 5.5, and Propositions 1.1–1.5 in *op. cit.* We repeat Jónsson's proofs for the convenience of the reader.

What is new in this section is (1) the realization that the same theorems and proofs go through for the case of completions (and perhaps other constructions as well) and (2) the use of the lemmas in the previous section (in particular, Lemmas 18 and 21) to prove a restricted form of Sahlqvist's preservation theorem.

From now on, assume that $\mathsf{K}$ is a class of Boolean algebras with completely additive operators (or, if the reader is so inclined, a more abstract class of structures as indicated above; some further remarks regarding this approach will be made at the end of the section). We will use the same symbol $f_\xi$ to denote a certain operator of an algebra in $\mathsf{K}$ and the corresponding operation symbol of the logical language associated with $\mathsf{K}$.

The following terminology is essentially from *op. cit.*, p. 474.

**Definition 4.** A term $\tau$ in $T$ is

(i) *monotone* (over $\mathsf{K}$) if $\tau^{\mathcal{A}}$ is monotone for all $\mathcal{A}$ in $\mathsf{K}$ .
(ii) *conservative* (over $\mathsf{K}$) if $\tau^{\mathcal{A}}$ is conservative for all $\mathcal{A}$ in $\mathsf{K}$ .
(iii) *expanding* (over $\mathsf{K}$) if $(\tau^{\mathcal{A}})^+ \leq \tau^{\mathcal{A}^+}$ for all $\mathcal{A}$ in $\mathsf{K}$.
(iv) *contracting* (over $\mathsf{K}$) if $\tau^{\mathcal{A}^+} \leq (\tau^{\mathcal{A}})^+$ for all $\mathcal{A}$ in $\mathsf{K}$.
(v) *stable* (over $\mathsf{K}$) if $\tau^{\mathcal{A}^+} = (\tau^{\mathcal{A}})^+$ for all $\mathcal{A}$ in $\mathsf{K}$.

The reference to the class $\mathsf{K}$ will usually be suppressed.

Our first goal is to show that the set $\Phi$ of monotone, stable, conservative terms includes the positive primitive terms and is closed under substitution. The terms $v_0 + v_1$ , $v_0 \cdot v_1$ , and $f_\xi(v_0, \ldots, v_{n_\xi - 1})$ for $\xi < \alpha$ are monotone since the operations they denote are completely additive. They are stable by Lemma 14 (applied to Boolean addition and multiplication) and the remark at the end of the preceding section (saying that the operation $g_\xi$ coincides with $f_\xi^+$). Lemma 15 (with $h = f_\xi$) and Lemma 16 ensure that they are conservative. Constant terms are trivially monotone, stable, and conservative. (More generally, terms denoting constant operations are trivially monotone, and they are conservative and stable by Lemma 14.) The variable $v_i$ is obviously monotone. It is stable by the assertion in Lemma 14 regarding projection operations. To see that it is conservative, suppose that $g_0, \ldots, g_{n-1}$ are monotone operations on $A$. Then

$$
\begin{aligned}
(v_i^{\mathcal{A}}[g_0, \ldots, g_{n-1}])^+ &= (g_i^{\mathcal{A}})^+ \\
&= v_i^{\mathcal{A}^+}[(g_0^{\mathcal{A}})^+, \ldots, (g_0^{\mathcal{A}})^+] \\
&= (v_i^{\mathcal{A}})^+[(g_0^{\mathcal{A}})^+, \ldots, (g_0^{\mathcal{A}})^+].
\end{aligned}
$$

The final equality uses the stability of the variable $v_i$. Thus, $\Phi$ contains all positive primitive terms.

The next three lemmas secure the closure of $\Phi$ under substitution. The proof of the first is easy and is left to the reader.

**Lemma 22.** *The collection of monotone operations on a set $A$ contains the projection functions and the constant functions, and is closed under functional composition.*

**Lemma 23.** *If $f$ is a monotone, conservative operation of rank $n$ on $A$ and if $g_0, \ldots, g_{n-1}$ are monotone, conservative operations of rank $m$ on $A$, then*

$$f[g_0, \ldots, g_{n-1}]$$

*is a conservative operation of rank $m$ on $A$.*

*Proof.* To prove the lemma let $k_0, \ldots, k_{m-1}$ be monotone operations on $A$ of rank $p$. We denote the sequences

$$(g_0, \ldots, g_{n-1}) \qquad \text{and} \qquad (k_0, \ldots, k_{m-1})$$

by $\bar{g}$ and $\bar{k}$ respectively. Then

$$\begin{aligned}\big((f[\bar{g}])[\bar{k}]\big)^+ &= (f[g_0[\bar{k}], \dots, g_{n-1}[\bar{k}])^+ \\ &= f^+[g_0[\bar{k}]^+, \dots, g_{n-1}[\bar{k}]^+] \\ &= f^+[{g_0}^+[{k_0}^+, \dots, {k_{m-1}}^+], \dots, {g_{n-1}}^+[{k_0}^+, \dots, {k_{m-1}}^+]] \\ &= (f^+[{g_0}^+, \dots, {g_{n-1}}^+])[{k_0}^+, \dots, {k_{m-1}}^+] \\ &= (f[g_0, \dots, g_{n-1}])^+[{k_0}^+, \dots, {k_{m-1}}^+]\end{aligned}$$

The first equality holds by definition of the composition $\big((f[\bar{g}])[\bar{k}]\big)$. The second holds because $f$ is assumed to be conservative and the compositions $g_0[\bar{k}], \dots, g_{n-1}[\bar{k}]$ are monotone operations by Lemma 22. The third equality holds because the operations $g_0, \dots, g_{n-1}$ are assumed to be conservative. The fourth holds by definition of the composition

$$(f^+[{g_0}^+, \dots, {g_{n-1}}^+])[{k_0}^+, \dots, {k_{m-1}}^+]\,.$$

The fifth holds because $f$ is assumed to be conservative and $g_0, \dots, g_{n-1}$ monotone. □

Since the notion of a complete extension of an operation $h$ is well behaved only when $h$ is monotone, the same applies to the induced operations $\tau^{\mathcal{A}}$. Therefore, in what follows *we shall suppose — without always restating it — that terms assumed to be expanding, contracting, stable, or conservative are also assumed to be monotone* (over $\mathsf{K}$). Of course, if a term has a certain concrete form (for example, if it is positive), then in proving that the term has a certain property like stability we must also prove that it is monotone; in such cases we will state this explicitly.

**Lemma 24.** *If $\tau(v_0, \dots, v_{n-1})$ is a stable, conservative term and if $\sigma_0, \dots, \sigma_{n-1}$ are stable terms, then $\tau(\sigma_0, \dots, \sigma_{n-1})$ is stable.*

*Proof.* Fix a structure $\mathcal{A}$ in $\mathsf{K}$. Then

$$\begin{aligned}\tau(\sigma_0, \dots, \sigma_{n-1})^{\mathcal{A}^+} &= \tau^{\mathcal{A}^+}[{\sigma_0}^{\mathcal{A}^+}, \dots, {\sigma_{n-1}}^{\mathcal{A}^+}] \\ &= (\tau^{\mathcal{A}})^+[({\sigma_0}^{\mathcal{A}})^+, \dots, ({\sigma_{n-1}}^{\mathcal{A}})^+] \\ &= (\tau^{\mathcal{A}}[{\sigma_0}^{\mathcal{A}}, \dots, {\sigma_{n-1}}^{\mathcal{A}}])^+ \\ &= \big(\tau(\sigma_0, \dots, \sigma_{n-1})^{\mathcal{A}}\big)^+\end{aligned}$$

The first and fourth equalities hold by the definition of the interpretation of a term in a structure. The second holds because all of the terms are assumed to be stable. The third holds because the term $\tau$ is assumed to be conservative and the terms $\sigma_0, \dots, \sigma_{n-1}$ are assumed to be monotone. (The term $\tau(\sigma_0, \dots, \sigma_{n-1})$ is monotone by Lemma 22.) □

The previous lemma continues to hold if we replace the term "stable" everywhere by the term "contracting" or everywhere by the term "expanding" (leaving the assumption that $\tau$ is conservative unchanged). The proof is nearly the same, but in the second step (where "=" must be replaced by "$\leq$" or by "$\geq$") we also make use of the assumed monotony of $\tau$.

The result established so far is summarized in the following theorem.

**Theorem 25.** *The collection of (monotone) stable, conservative terms contains the positive primitive terms and is closed under substitution.*

The following corollary is consequence of the previous theorem and the characterization of strictly positive terms that was given just before Theorem 6.

**Corollary 26.** *Every strictly positive term is monotone, stable, and conservative.*

Our next goal is to show that the set $\Phi$ of (monotone) expanding terms includes all positive terms. The proof is similar in character to the proof of Theorem 25, but uses the characterization of positive terms that was given in Theorem 6. The following lemma ensures the closure of $\Phi$ under substitution.

**Lemma 27.** *The collection of expanding terms is closed under substitution.*

*Proof.* Suppose that $\tau(v_0, \ldots, v_{n-1})$ and $\sigma_0, \ldots, \sigma_{n-1}$ are expanding. Then

$$\begin{aligned}(\tau(\sigma_0, \ldots, \sigma_{n-1})^{\mathcal{A}})^+ &= (\tau^{\mathcal{A}}[\sigma_0^{\mathcal{A}}, \ldots, \sigma_{n-1}^{\mathcal{A}}])^+ \\ &\leq (\tau^{\mathcal{A}})^+[(\sigma_0^{\mathcal{A}})^+, \ldots, (\sigma_{n-1}^{\mathcal{A}})^+] \\ &\leq (\tau^{\mathcal{A}})^+[\sigma_0^{\mathcal{A}^+}, \ldots, \sigma_{n-1}^{\mathcal{A}^+}] \\ &\leq \tau^{\mathcal{A}^+}[\sigma_0^{\mathcal{A}^+}, \ldots, \sigma_{n-1}^{\mathcal{A}^+}] \\ &= \tau(\sigma_0, \ldots, \sigma_{n-1})^{\mathcal{A}^+}\end{aligned}$$

The first and last steps follow from the definition of the interpretation of a term in a structure. The second step follows from Lemma 15 and the assumption that the terms are monotone. For the next step, we use the assumption that the terms $\sigma_i$ are expanding, and also the fact that the operation $(\tau^{\mathcal{A}})^+$ must be monotone (this follows from Lemma 11, since $\tau^{\mathcal{A}}$ is assumed to be monotone). The fourth step uses the assumption that $\tau$ is expanding.

Thus, $\tau(\sigma_0, \ldots, \sigma_{n-1})$ is expanding. It is monotone by Lemma 22. $\square$

**Lemma 28.** *If $\tau$ is a stable term, then $\tau^d$ is (monotone) expanding.*

*Proof.* By assumption, $\tau^{\mathcal{A}}$ is monotone. Therefore, so is $(\tau^d)^{\mathcal{A}}$, by Lemma 17. Hence,

$$((\tau^d)^{\mathcal{A}})^+ = ((\tau^{\mathcal{A}})^d)^+ \leq ((\tau^{\mathcal{A}})^+)^d = (\tau^{\mathcal{A}^+})^d = (\tau^d)^{\mathcal{A}^+}.$$

The first and last equalities use Lemma 8; the inequality follows from Lemma 18 and the monotonicity of $\tau$; and the next equality holds because $\tau$ is stable. $\square$

**Theorem 29.** *Every positive term is monotone and expanding.*

*Proof.* Let $\Phi$ be the set of (monotone) expanding terms. If a term is positive primitive, then it is monotone and stable, by Theorem 25. Hence, it is expanding by definition, and its dual is monotone and expanding by Lemma 28. This shows that the positive primitive terms and their duals are in $\Phi$. Since $\Phi$ is closed under substitution, by Lemma 27, it follows that the smallest set of terms containing the positive primitive terms and their duals and closed under substitution is included in $\Phi$. Therefore $\Phi$ includes all positive terms, by Theorem 6. $\square$

We are now in a position to prove one of two preservations theorems. A formula is said to be *preserved under completions* (over $\mathsf{K}$) if its validity in an algebra $\mathcal{A}$ of $\mathsf{K}$ implies its validity in the completion $\mathcal{A}^+$. A formula is *strictly positive* if it contains no occurrences of the (logical) negation symbol.

**Theorem 30** (First Preservation Theorem)**.** (i) *An equation $\sigma \approx \tau$ is preserved under completions whenever $\sigma$ and $\tau$ are stable terms.*

(ii) *More generally, an implication* $\varphi \to \sigma \approx \tau$ *is preserved under completions whenever* $\sigma$ *and* $\tau$ *are stable terms and* $\varphi$ *is a Boolean combination of equations of the form* $\rho \approx 0$ *with* $\rho$ *stable.*

(iii) *An inequality* $\sigma \lesssim \tau$ *is preserved under completions whenever* $\sigma$ *is a contracting term and* $\tau$ *is an expanding term.*

(iv) *More generally, an implication* $\varphi \to \sigma \lesssim \tau$ *is preserved under completions whenever* $\sigma$ *is a contracting term,* $\tau$ *is an expanding term, and* $\varphi$ *is a strictly positive Boolean combination of equations of the form* $\rho \approx 0$ *with* $\rho$ *expanding.*

*Proof.* The proof of (i) is easy. If the equation $\sigma \approx \tau$ is valid in $\mathcal{A}$, then $\sigma^{\mathcal{A}} = \tau^{\mathcal{A}}$, and therefore $(\sigma^{\mathcal{A}})^+ = (\tau^{\mathcal{A}})^+$. By stability $\sigma^{\mathcal{A}^+} = \tau^{\mathcal{A}^+}$; hence $\sigma \approx \tau$ is also valid in $\mathcal{A}^+$.

The proof of (iii) is quite similar, but uses the definitions of expanding and contracting instead of the definition of stability.

We illustrate the proof of (ii) with some examples. Expand $\mathcal{A}$ by adjoining its unary discriminator $k$ as an additional operator. Consider, as a first example, the implication

$$\rho \approx 0 \to \sigma \approx \tau \,. \tag{1}$$

It is valid in either $\mathcal{A}$ or $\mathcal{A}^+$ if and only if the equation

$$\sigma + k(\rho) = \tau + k(\rho) \tag{2}$$

is valid in the corresponding expanded algebra. The terms $\rho$, $\sigma$, and $\tau$ are stable by assumption, and the terms $v_0 + v_1$ and $k(v_0)$ are positive primitive and hence stable by Theorem 25 (here we use the fact that the operation $k$ is completely additive). It follows that the terms on both the right-hand side and the left-hand side of (2) are stable. Thus, (2) is preserved under completions (by part (i)), and hence so is (1).

As a second example, consider the implication

$$(\rho_0 \approx 0 \wedge \rho_1 \approx 0 \wedge \rho_2 \not\approx 0 \wedge \rho_3 \not\approx 0) \to \sigma \approx \tau \,. \tag{3}$$

It is valid in either $\mathcal{A}$ or $\mathcal{A}^+$ if and only if the equation

$$[\sigma \cdot k(\rho_2) \cdot k(\rho_3) + k(\rho_0) + k(\rho_1)] \approx [\tau \cdot k(\rho_2) \cdot k(\rho_3) + k(\rho_0) + k(\rho_1)] \tag{4}$$

is valid in the corresponding expanded algebra. As in the case of (2), the terms on both the right-hand side and the left-hand side of (4) are stable. Thus, (4) is preserved under completions (by part (i)), and hence so is (3).

The previous example shows how to handle the case when the formula $\varphi$ in the implication

$$\varphi \to \sigma \approx \tau \tag{5}$$

is a conjunction of equations and negations of equations of the specified form. Suppose, next, that $\varphi$ is a disjunction of formulas $\varphi_i$ (for $i < n$) that are conjunctions of equations and negations of equations as specified. Then the implication (5) is logically equivalent to the conjunction of the implications $\varphi_i \to \sigma \approx \tau$. We have just seen that each of the latter formulas is preserved under completions. Hence, so is their conjunction, and therefore also (5). Finally, if $\varphi$ is any Boolean combination of equations of the specified form, then it is logically equivalent to a disjunction

$\psi$ of conjunctions of such equations and their negations. Since, as was shown, the implication $\psi \to \sigma \approx \tau$ is preserved under completions, so is (5).

Part (iv) is a consequence of (iii), and its proof is similar to that of (ii). For example, to prove that the implication

$$(\rho_0 \approx 0 \wedge \rho_1 \approx 0) \to \sigma \lesssim \tau$$

is preserved under completions, use the fact that it is valid in either $\mathcal{A}$ or $\mathcal{A}^+$ just in case the equation

$$\sigma \lesssim [\tau + k(\rho_0) + k(\rho_1)]$$

is valid in the corresponding expanded algebra. Since the term $v_0 + k(v_1) + k(v_2)$ is strictly positive, it is expanding (and, in fact, stable) by Corollary 26. Therefore the substitution instance $\tau + k(\rho_0) + k(\rho_1)$ is expanding by Lemma 27. Because the term $\sigma$ is contracting by assumption, part (iii) applies. The general case is handled just as in the preceding paragraph, using the fact that every positive quantifier-free formula is logically equivalent to a disjunction of conjunctions of atomic formulas. □

Parts (ii) and (iv) of the preceding theorem also hold for canonical extensions. They generalize Propositions 1.2 and 1.4 in Jónsson [1994].

The following is an immediate corollary of the previous theorem and Theorems 25 and 29. Part (i) is due to Monk [1970]; a weaker version of part (ii), in which $\varphi$ is assumed to be either a conjunction or else a disjunction of equations $\rho \approx 0$ and their negations (with $\rho$ strictly positive), is stated there without proof. For the proof of part (v), observe that the term 1 is stable, and hence contracting. Therefore, the inequality $1 \lesssim \tau$ is preserved under completions. The reverse inequality is universally valid.

**Corollary 31.** (i) *An equation $\sigma \approx \tau$ is preserved under completions whenever $\sigma$ and $\tau$ are strictly positive terms.*

(ii) *More generally, an implication $\varphi \to \sigma \approx \tau$ is preserved under completions whenever $\sigma$ and $\tau$ are strictly positive terms and $\varphi$ is a Boolean combination of equations of the form $\rho \approx 0$ with $\rho$ strictly positive.*

(iii) *An inequality $\sigma \lesssim \tau$ is preserved under completions whenever $\sigma$ is a strictly positive term and $\tau$ is a positive term.*

(iv) *More generally, an implication $\varphi \to \sigma \lesssim \tau$ is preserved under completions whenever $\sigma$ is a strictly positive term, $\tau$ is a positive term, and $\varphi$ is a strictly positive Boolean combination of equations of the form $\rho \approx 0$ with $\rho$ positive.*

(v) *The equation $\tau \approx 1$ is preserved under completions whenever $\tau$ is a positive term.*

**Theorem 32** (Second Preservation Theorem)**.** *Suppose that $\sigma_0, \ldots, \sigma_{n-1}$ are contracting terms, $\tau_0, \ldots, \tau_{m-1}$ expanding terms, and $\rho$ a contracting and conservative term. Then the identitity*

$$\rho(\sigma_0, \ldots, \sigma_{n-1}, -\tau_0, \ldots, -\tau_{m-1}) \approx 0 \tag{1}$$

*is preserved under completions.*

*Proof.* Let $u_0, \ldots, u_{m-1}$ be distinct variables that do not occur in (1) and that are distinct from the variables $v_0, \ldots, v_{m-1}$. The term $v_0 \cdot u_0 + \cdots + v_{m-1} \cdot u_{m-1}$ is strictly positive and hence monotone expanding (in fact, stable) by Corollary 26. Therefore, the substitution instance

$$\gamma = \tau_0 \cdot u_0 + \cdots + \tau_{m-1} \cdot u_{m-1}$$

is expanding, by Lemma 27. The term

$$\delta = \rho(\sigma_0, \ldots, \sigma_{n-1}, u_0, \ldots, u_{m-1})$$

is contracting, by the remark following Lemma 24 (the terms $\sigma_i$ are assumed to be contracting, the variables $u_i$ are strictly positive and therefore stable by Corollary 26, and the term $\rho$ is assumed to be contracting and conservative). It is not difficult to check that the validity of (1) is equivalent to the validity of the implication

$$\gamma \approx 0 \to \delta \lesssim 0 \tag{2}$$

in any algebra. (Indeed, suppose that (2) holds. Substituting the term $-\tau_i$ for $u_i$ (for each $i < m$) in (2) we obtain a substitution instance of (2) whose antecent is trivially true. Therefore the consequent must be true, and the consequent is obviously equivalent to (1). Now suppose that (1) is valid, and assume that the antecedent of (2) holds. Then $u_i \lesssim -\tau_i$ must hold. Therefore

$$\delta \lesssim \rho(\sigma_0, \ldots, \sigma_{n-1}, -\tau_0, \ldots, -\tau_{m-1})$$

holds, since $\rho$ is monotone. If the right-hand side of this inequality is equivalent to 0, then obviously so is the left-hand side.) Since the implication (2) is preserved under completions, by Theorem 30(iv), so is the equation (1). □

We shall call a unary term $\tau$ *conjugated* (over $\mathsf{K}$) if it denotes a conjugated (unary) operation in each structure in $\mathsf{K}$. We do not require the operation that is conjugate to the operation $\tau^{\mathcal{A}}$ (for $\mathcal{A}$ in $\mathsf{K}$) to be denotable by a term.

**Lemma 33.** *The dual of a stable, conjugated term is stable.*

*Proof.* Let $\tau$ be a unary term. Then

$$(\tau^d)^{\mathcal{A}^+} = (\tau^{\mathcal{A}^+})^d = ((\tau^{\mathcal{A}})^+)^d = ((\tau^{\mathcal{A}})^d)^+ = ((\tau^d)^{\mathcal{A}})^+ .$$

The first and last equalities follow from Lemma 8, and the second from the assumption that $\tau$ is stable. The third equality (and this is the crucial one) follows from Lemmas 18 and 21, and from the assumption that $\tau$ is conjugated. □

It should be emphasized that the conjugated term in the preceding lemma is assumed to be unary; in other words, it has just one variable. The lemma does not extend to terms with more than one variable, as an example in the next section shows.

**Definition 5.** (i) A *generalized Sahlqvist term* is a term of the form

$$\rho(\sigma_0^d, \ldots, \sigma_{n-1}^d, -\tau_0, \ldots, -\tau_{m-1}),$$

where $\rho$ is a strictly positive term, $\sigma_0, \ldots, \sigma_{n-1}$ are strictly positive unary terms, and $\tau_0, \ldots, \tau_{m-1}$ are positive terms.

(ii) A *simple Sahlqvist term* is a term of the form

$$\rho(\sigma_0, \ldots, \sigma_{n-1}, -\tau_0, \ldots, -\tau_{m-1}),$$

where $\rho$ and $\sigma_0, \ldots, \sigma_{n-1}$ are strictly positive (but not necessarily unary) terms and $\tau_0, \ldots, \tau_{m-1}$ are positive terms.

(iii) A *generalized Sahlqvist equation* (respectively, a *simple Sahlqvist equation*) is an equation of the form $\tau \approx 0$, where $\tau$ is a generalized Sahlqvist term (respectively, a simple Sahlqvist term).

(iv) A *generalized Sahlqvist inequality* (respectively, a *simple Sahlqvist inequality*) is an inequality of the form $\sigma \lesssim \tau$, where $\sigma$ is a generalized Sahlqvist term (respectively, a simple Sahlqvist term) and $\tau$ is positive.

In the definition of a simple Sahlqvist term there is no loss of generality in assuming that the terms $\sigma_i$ are actually variables, since the designated strictly positive terms may be viewed as part of $\rho$. For example, the simple Sahlqvist term

$$f_\xi(v_0 + v_2 \, , v_1 \cdot v_3 \, , -(v_1 + v_4))$$

can be thought of as having the form $\rho(\sigma_0 \, , \sigma_1 \, , -\tau)$, where $\rho$ is the term $f_\xi(v_0 \, , v_1 \, , v_2)$, $\sigma_0$ is $v_0 + v_2$ , $\sigma_1$ is $v_1 \cdot v_3$ , and $\tau$ is $v_1 + v_4$ . Alternatively, it can be thought of as having the form $\rho(v_0 \, , v_1 \, , v_2 \, , v_3 \, , -\tau)$, where $\rho$ is the term $f_\xi(v_0 + v_2 \, , v_1 \cdot v_3 \, , v_4)$ and $\tau$ is as before; here the terms $\sigma_0$ and $\sigma_1$ defined above are being viewed as part of the term $\rho$ . It follows that any simple Sahlqvist term may be thought of as a generalized Sahlqvist term in which the terms $\sigma_i$ are just variables.

**Corollary 34.** (i) *Every simple Sahlqvist equation and inequality is preserved under completions.*

(ii) *Every generalized Sahlqvist equation and inequality in which the designated strictly positive unary terms are conjugated* (*over* $\mathsf{K}$) *is preserved under completions.*

*Proof.* To prove the first assertion in (ii), suppose that

$$\rho(\sigma_0^d, \ldots, \sigma_{n-1}^d, -\tau_0, \ldots, -\tau_{m-1}) \, ,$$

is a Sahlqvist term in which the terms $\sigma_0, \ldots, \sigma_{n-1}$ are strictly positive, conjugated, and unary. Then these latter terms are stable by Corollary 26, and hence so are their duals, by Lemma 33. The terms $\tau_0, \ldots, \tau_{m-1}$ are positive and therefore expanding, by Theorem 29. The term $\rho$ is strictly positive and therefore conservative and stable, by Corollary 26. The assertion now follows directly from the second preservation theorem.

The second assertion of (ii) follows from the first. A generalized Sahlqvist inequality $\sigma \lesssim \tau$ is equivalent to the generalized Sahlqvist equation $\sigma \cdot -\tau \approx 0$, and any condition on the designated unary terms of $\sigma$ transfers from the inequality to the equation.

The proof of part (i) is quite similar, but uses Corollary 26 instead of Lemma 33. We leave the details to the reader. □

In the usual definition of a Sahlqvist term, the terms $\sigma_i^d$ are replaced by terms of the form

$$f_0^d(f_1^d(\ldots f_{n-1}^d(v)\ldots)) \, , \tag{1}$$

where $f_0, \ldots, f_{n-1}$ are basic operation symbols of rank 1 that are different from negation. The term in (1) is obviously equivalent (by the law of double negation) to the dual of the strictly positive unary term

$$f_0(f_1(\ldots f_{n-1}(v)\ldots)) \, . \tag{2}$$

If the operations denoted by $f_0, \ldots, f_{n-1}$ are conjugated, then so is the operation denoted by the term in (2); indeed, if $g_i$ is the conjugate of $h_i$ for $i < n$, then the composition of $g_{n-1}, g_{n-2}, \ldots, g_0$ (in that order) is the conjugate of the composition of $h_0, h_1, \ldots, h_{n-1}$. Thus, the hypotheses of Corollary 34(ii) are satisfied, in particular, when in the Sahlqvist term

$$\rho(\gamma_0, \ldots, \gamma_{n-1}, -\tau_0, \ldots, -\tau_{m-1})$$

the terms $\gamma_i$ have the form (1) and the operations denoted by the symbols $f_i$ are conjugated in each structure in $\mathsf{K}$.

Lemma 22 through Theorem 29, and parts (i) and (ii) of the First Preservation Theorem and its corollary, are valid in the context of abstract classes of structures satisfying the conditions set forth in the first paragraph of this section. The remaining results in the section require the additional assumption that, among the fundamental operations of the structures, there are two completely additive binary operations $+$ and $\cdot$, and two distinguished elements 0 and 1, satisfying the following conditions for all elements $x, y$ in each structure:

$$\begin{gathered} x + 0 = 0 + x = x \quad , \quad x \cdot 0 = 0 \cdot x = 0\,, \\ x + 1 = 1 + x = 1 \quad , \quad x \cdot 1 = 1 \cdot x = x\,, \\ x + y = 0 \quad \text{iff} \quad x = 0 \text{ and } y = 0\,, \\ x \cdot y = 0 \quad \text{iff} \quad x \leq -y\,. \end{gathered}$$

Specifically, the properties expressed in the first two lines of the displayed equations are need to prove the remaining parts of the First Preservation Theorem and its corollary. The other requirements, as well as the reflexivity of $\leq$, are needed to derive the Second Preservation Theorem and Corollary 34. Finally, the proofs of Lemma 33 and Corollary 34 require the addition of Lemma 21 to the list of basic assumptions about the properties of the extension construction.

## 4. Equations that are not preserved under completions

We now give, in algebraic form, an example from Venema [1993] of a Sahlqvist equation that is not preserved under completions. In this example, the term $f(v)$ in equation (1) below (the designated strictly positive unary term in the formulation of Corollary 34(ii)) denotes an operation that is completely additive, but not conjugated. Thus, the assumption that these terms are conjugated cannot be weakened to the assumption that they are complete additive.

Take $S$ to be the set of ordinals less than or equal to $\omega + 2$ (where $\omega$ is the first infinite ordinal) and take $R$ to be the binary relation on $S$ consisting of the (ordered) pairs $(0, n)$ for $n < \omega$, the pairs $(2n, \omega)$ for $0 < n < \omega$, the pairs $(2n + 1, \omega + 1)$ for $n < \omega$, and the pairs $(\omega, \omega + 2)$, $(\omega + 1, \omega + 2)$ (see Figure 1). Let

$$\mathcal{B} = \langle B\,, \cup\,, \sim\,, h\,\rangle$$

be the complex algebra of the relational structure $\langle S\,, R\,\rangle$. Thus, the elements of $B$ are just the subsets of $S$, and $h$ is a completely additive unary operation on $B$ that is defined on singletons by the rule

$$h(\{x\}) = \{y : (x, y) \text{ is in } R\}\,.$$

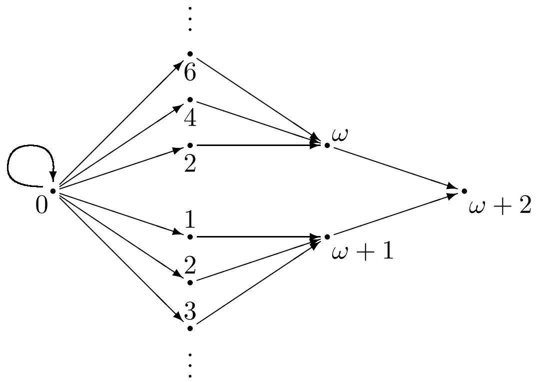


FIGURE 1. The binary relation $R$.

It is easy to compute that

$$\begin{aligned}
&h(\{0\}) = \{n : n < \omega\},\\
&h(\{2n\}) = \{\omega\} \qquad \text{for } 0 < n < \omega,\\
&h(\{2n+1\}) = \{\omega+1\} \qquad \text{for } n < \omega,\\
&h(\{\omega\}) = h(\{\omega+1\}) = \{\omega+2\},\\
&h(\{\omega+2\}) = \varnothing.
\end{aligned}$$

From these calculations it follows that, for any subset $X$ of $S$, the set $h(X)$ is finite (in fact, it is a subset of $\{\omega, \omega+1, \omega+2\}$) if $0 \notin X$, and it is cofinite (in fact, it contains every natural number) if $0 \in X$. Thus, the collection $A$ of all finite and cofinite subsets of $S$ is closed under the operation $h$. Let $g$ denote the restriction of $h$ to $A$. Then the structure

$$\mathcal{A} = \langle A\,,\, \cup\;,\; \sim\;, g\,\rangle$$

is an atomic Boolean algebra with a completely additive unary operator $g$, and it is easy to check that $\mathcal{B}$ is the completion of $\mathcal{A}$. In particular, $h = g^{+}$.

Let $f$ denote a unary operation symbol that is interpreted as $h$ in $\mathcal{B}$ and as $g$ in $\mathcal{A}$. We shall show that the (positive) Sahlqvist equation

$$f(f^{d}(v)) \lesssim f^{d}(f(v)) \tag{1}$$

is valid in $\mathcal{A}$, but fails to be valid in the completion $\mathcal{B}$. (Strictly speaking, (1) is not an equation. However, it is well known that, in a theory of Boolean algebras with operators, every inequality is equivalent to an equation.)

We begin by showing that (1) is valid in $\mathcal{A}$. Let $X$ be an arbitrary finite or cofinite subset of $S$, and suppose that $x$ is any element of $S$. We must show that if $x$ is in the set

$$g(g^{d}(X)) = g({\sim}g({\sim}X))\,,$$

then it is in the set

$$g^{d}(g(X)) = {\sim}g({\sim}g(X))\,.$$

Consider, first, the case when the element $x = \omega+2$ is in $g({\sim}g({\sim}X))$. From the definition of $g$ (see the definition of $h$) it follows that at least one of the elements $\omega$

and $\omega+1$ must be in the set $\sim g(\sim X)$, since these are the only two elements that are mapped to $\omega+2$ by $g$; suppose that $\omega$ is in the set (the case when $\omega+1$ is in the set is completely analogous). Then $\omega$ is not in $g(\sim X)$, so none of the elements $2n$, for $0 < n < \omega$, can be in $\sim X$ (each of these elements is mapped to $\omega$ by $g$). Therefore, they must all be in $X$. This forces $X$ to be infinite and hence cofinite. It follows that $2n+1$ is in $X$ for some (and, actually, for almost all) $n < \omega$. Hence, $\omega+1$ is in $g(X)$; also, $\omega$ is in $g(X)$, since, e.g., 2 is in $X$. As a consequence, neither of $\omega$, $\omega+1$ is in $\sim g(X)$, so $\omega+2$ cannot be in $g(\sim g(X))$. This means that $\omega+2$ must be in $\sim g(\sim g(X))$, as desired.

Now consider the case when $x$ is one of the elements less than $\omega+2$. A straightforward computation, using the definition of $g$, establishes the following chain of equivalences for any subset $Y$ of $S$:

$$\begin{aligned} 0 \in Y \quad &\text{iff} \quad \{n : n < \omega\} \subseteq g(Y) \\ &\text{iff} \quad \sim g(Y) \subseteq \{\omega, \omega+1, \omega+2\} \\ &\text{iff} \quad g(\sim g(Y)) \subseteq \{\omega+2\}. \end{aligned}$$

Suppose that $x$ is in the set $g(\sim g(\sim X))$; then this set contains an element different from $\omega+2$ (namely $x$). The above chain of equivalences (with $Y = \sim X$) shows that 0 cannot be in $\sim X$; hence, it is in $X$. Applying again the above equivalences (with $Y = X$), we see that the set $g(\sim g(X))$ contains at most the element $\omega+2$. In particular, it does not contain $x$, so $x$ is in $\sim g(\sim g(X))$.

This completes the verification that equation (1) holds in $\mathcal{A}$. To show that it fails in $\mathcal{B}$, let $X$ be the set of positive, even integers. Then $\sim X$ is the set of odd integers, together with 0, $\omega$, $\omega+1$, and $\omega+2$. Successive computations give:

$$h(\sim X) = S \sim \{\omega\} \quad , \quad \sim h(\sim X) = \{\omega\} \quad , \quad h(\sim h(\sim X)) = \{\omega+2\}\,.$$

On the other hand,

$$h(X) = \{\omega\} \quad , \quad \sim h(X) = S \sim \{\omega\} \quad , \quad h(\sim h(X)) = S \quad , \quad \sim h(\sim h(X)) = \varnothing\,.$$

Therefore, $h(\sim h(\sim X))$ is not included in $\sim h(\sim h(X))$.

The following remarks are intended to illuminate some of the intuitions underlying the preceding construction. The argument in the penultimate paragraph above (starting with "Now consider") actually shows that for *any* subset $X$ of $S$, each element of $h(\sim h(\sim X))$ — except possibly $\omega+2$ — must be in $\sim h(\sim h(X))$. The basic reason for this is the following characterization of the elements $x$ of $S$ that are different from $\omega+2$: any two predecessors $u, v$ of $x$ (in $R$) have a common predecessor $y$ (namely 0). If $x$ were in $h(\sim h(\sim X))$ and also in $h(\sim h(X))$, then it would have a predecessor $u$ in $\sim h(\sim X)$ and a predecessor $v$ in $\sim h(X)$. In which set — $X$ or $\sim X$ — would the common predecessor $y$ of $u$ and $v$ be? If it were in $X$, then $v$ would be in $h(X)$, and if it were in $\sim X$, then $u$ would be in $h(\sim X)$. Both of these conclusions lead, of course, to contradictions.

The distinguishing feature of $\omega+2$ is the fact that it has two predecessors, $\omega$ and $\omega+1$, each of which has an infinite family of predecessors, and none of these infinitely many predecessors is a common predecessor of both $\omega$ and $\omega+1$. If $\omega+2$ is in $h(\sim h(\sim X))$, then one of its two predecessor is not in $h(\sim X)$, so none of the infinitely many predecessors of that predecessor is in $\sim X$. In the case when we admit only finite and cofinite sets, this forces $X$ to be cofinite, so that both predecessors of $\omega+2$ must be in $h(X)$; hence $\omega+2$ must be in $\sim h(\sim h(X))$. However, in the case when all infinite subsets of $S$ are admitted, we can choose $X$ to be the set of predecessors of just one of the predecessors of $\omega+2$. Because the two predecessors of $\omega+2$ do not have a common predecessor, this forces one of them to be in $h(X)$ and the other one to be in $\sim h(X)$, causing $\sim h(\sim h(X))$ to be empty.

Is it necessary, in Corollary 34(ii), to assume that the designated strictly positive terms (the terms $\sigma_0, \ldots \sigma_{n-1}$ in the definition of a Sahlqvist term) are in fact unary? In other words, does the assertion continue to hold when these terms denote operations of rank $>1$ that are conjugated in each coordinate? The next example shows that this need not be the case, even when the operations have rank 2, and even when they are their own conjugates in each coordinate.

Take $S$ to be the set of ordinals less than or equal to $\omega$ and take $R$ to be the ternary relation on $S$ consisting of the (ordered) triples $(n, n, n)$ for $n<\omega$ and all permutations of the triples $(2n, 2n+1, \omega)$ for $n<\omega$. (For instance, the triples $(\omega, 2n, 2n+1)$ and $(2n+1, \omega, 2n)$ are in $R$ for all $n<\omega$.) The relation $R$ is *totally symmetric* in the sense that every permutation of a triple in $R$ is again in $R$.

Let
$$\mathcal{B}=\langle B\,,\,\cup\,,\,\sim\,,h\,\rangle$$
be the complex algebra of the structure $\langle S\,, R\,\rangle$. As in the previous example, the elements of $B$ are just the subsets of $S$, and $h$ is a completely additive binary operation on $B$ that is defined on pairs of singletons by the rule
$$h(\{x\},\{y\})=\{z:(x, y, z) \text{ is in } R\}\,.$$
For all $n<\omega$ we have
$$\begin{aligned}
&h(\{n\},\{n\})=\{n\}, \\
&h(\{2n\},\{2n+1\})=h(\{2n+1\},\{2n\})=\{\omega\}, \\
&h(\{2n\},\{\omega\})=h(\{\omega\},\{2n\})=\{2n+1\}, \\
&h(\{2n+1\},\{\omega\})=h(\{\omega\},\{2n+1\})=\{2n\},
\end{aligned}$$
and
$$h(\{x\},\{y\})=\varnothing$$
for all other pairs of elements $x, y$ from $S$.

The operation $h$ is *self-conjugate* in the sense that it is its own left- and right-conjugate:
$$h(X, Y) \cap Z=\varnothing \quad \text{iff} \quad h(Z, Y) \cap X=\varnothing \quad \text{iff} \quad h(X, Z) \cap Y=\varnothing$$
for all subsets $X, Y, Z$ of $S$. Indeed, it is easy to check that the complex operation derived from a totally symmetric relation is always self-conjugate (no matter what the rank of the relation is).

The set $A$ of finite and cofinite subsets of $S$ is closed under the operation $h$. To see this, suppose that $X$ and $Y$ are subsets of $S$. If both of them are finite, then the previous computations show that $h(X, Y)$ is also finite. If just one of them is finite, but does not contain $\omega$, then again $h(X, Y)$ is finite. If one of them is cofinite — say it includes the set $C_k=\{n: k \leq n<\omega\}$ for some $k<\omega$ — and the other contains the element $\omega$, then $h(X, Y)$ is cofinite, since it must contain $C_{k+1}$.

Finally, if both sets are cofinite, then they both contain $C_k$ for some $k < \omega$, so of course the same is true of $h(X, Y)$.

Take $g$ to be the restriction of $h$ to $A$. Then the structure

$$\mathcal{A} = \langle A\,,\,\cup\,,\,\sim\,,g\,\rangle$$

is an atomic Boolean algebra with a conjugated binary operator $g$, and $\mathcal{B}$ is the completion of $\mathcal{A}$.

Suppose that $f$ is a binary operation symbol that denotes the operation $h$ in $\mathcal{B}$ and the operation $g$ in $\mathcal{A}$. We shall show that the (positive) Sahlqvist equation

$$f^d(u, v) \lesssim f(u, u) + f(v, v) \tag{2}$$

is valid in $\mathcal{A}$, but not in its completion $\mathcal{B}$.

To show that (2) is valid in $\mathcal{A}$, let $X$ and $Y$ be finite or cofinite subsets of $S$, and consider an arbitrary element $x$ in $\sim g(\sim X, \sim Y)$. Then $x$ is not in $g(\sim X, \sim Y)$. In case $x$ is a natural number, this means that $x$ cannot be in both $\sim X$ and $\sim Y$ (since $g(\{x\}, \{x\}) = \{x\}$); consequently, $x$ must either be in $X$ or else in $Y$, and hence in $g(X, X)$ or in $g(Y, Y)$. In case $x = \omega$, it means that for no natural number $n$ do we have both $2n$ in $\sim X$ and $2n + 1$ in $\sim Y$ (since $g(\{2n\}, \{2n + 1\}) = \{\omega\}$); therefore, the two sets $\sim X$ and $\sim Y$ cannot both be cofinite, or, put another way, either $X$ or $Y$ is cofinite. Suppose $X$ is cofinite. Then for some natural number $n$ we have $2n$ and $2n + 1$ in $X$, so $\omega$ is in $g(X, X)$. The case when $Y$ is cofinite is completely analogous.

This completes the proof that equation (2) is valid in $\mathcal{A}$. To see that it fails in $\mathcal{B}$, take $X$ and $Y$ both to be the set of even (natural) numbers. Then $\sim X$ and $\sim Y$ are both the set of odd numbers, together with $\omega$. A simple computation shows that $h(\sim X, \sim Y)$ is precisely the set of all natural numbers. (It contains the even numbers because $h(\{2n+1\}, \{\omega\}) = \{2n\}$, and it contains the odd numbers because $h(\{2n+1\}, \{2n+1\}) = \{2n+1\}$.) Therefore, $\sim h(\sim X, \sim Y)$ consists of just $\omega$. But $h(X, X)$ and $h(Y, Y)$ are both the set of even numbers. Thus, $\sim h(\sim X, \sim Y)$ is not included in their union.

We have just seen that equation (2) is not preserved under completions. Since the term on the right-hand side of the equation is stable (it is strictly positive), the term on the left-hand side cannot be stable, by part (iii) of the First Preservation Theorem. This shows that the dual of a stable, conjugated term need not be stable. In other words, Lemma 33 does not generalize to conjugated terms of higher rank. The proof of that lemma depends essentially on Lemmas 18 and 21. Moreover, Lemma 18 holds for arbitrary monotone operations, and in particular for conjugated operations of arbitrary rank. Thus, Lemma 21 must fail for conjugated operations of higher rank.

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