MacNeille completions of lattice expansions

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ABSTRACT. There are two natural ways to extend an arbitrary map between (the carriers of) two lattices, to a map between their MacNeille completions. In this paper we investigate which properties of lattice maps are preserved under these constructions, and for which kind of maps the two extensions coincide.

Our perspective involves a number of topologies on lattice completions, including the Scott topologies and topologies that are induced by the original lattice. We provide a characterization of the MacNeille completion in terms of these induced topologies.

We then turn to expansions of lattices with additional operations, and address the question which equational properties of such lattice expansions are preserved under various types of MacNeille completions that can be defined for these algebras. For a number of cases, including modal algebras and residuated (ortho)lattice expansions, we provide reasonably sharp sufficient conditions on the syntactic shape of equations that guarantee preservation. Generally, our results show that the more residuation properties the primitive operations satisfy, the more equations are preserved.

1. Introduction

DEDEKIND's construction [11] of the reals as so-called cuts of the rationals, was generalized by MACNEILLE [35] to a method for embedding an arbitrary poset $\mathbb{P}$ into a complete lattice $\overline{\mathbb{P}}$. This completion has received a lot of attention in the literature on lattices and order, and goes under various names, including the (Dedekind-)MacNeille completion, the normal completion, and the completion by cuts. It has the nice property of preserving all existing meets and joins of the original partial order $\mathbb{P}$; in particular, if $\mathbb{P}$ is a lattice, then it is a sublattice of its MacNeille completion. Various abstract characterizations are possible; for instance, both BANASCHEWSKI [4] and SCHMIDT [42] showed that $\overline{\mathbb{P}}$ is determined as the extension of $\mathbb{P}$, unique modulo isomorphisms that fix $\mathbb{P}$, in which $\mathbb{P}$ is both meet- and join dense. In this paper we will take this characterization as the *definition* of the MacNeille completion (see section 2 for the details).

It is natural to ask, which properties of a lattice are *MacNeille-canonical*, that is, preserved under taking MacNeille completions. It is well known that the construction is not very well behaved when it comes to preserving equational properties.

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In particular, FUNAYAMA [17] provided an example of a distributive lattice whose completion is not distributive. A much stronger negative result, due to HARDING [27], states that the only MacNeille-canonical varieties are the trivial variety and the variety of all lattices. On the other hand, there are positive results as well. For instance, already MACNEILLE [35] proved that his construction, when applied to (the lattice reduct of) a Boolean algebra, yields again a Boolean algebra; and the same holds for Heyting algebras, see for instance BALBES & DWINGER [3]. For some properties of complete lattices, including Scott continuity, complete distributivity and algebraicity, one may isolate a corresponding property of partial orders, such that an arbitrary partial order has the corresponding property iff its MacNeille completion satisfies the original one; for details we refer to ERNÉ [15] and references therein.

Recall however, that Dedekind's concern was not only the lattice- and order-theoretic properties of $\mathbb{Q}$ and $\mathbb{R}$. He also extended the basic arithmetical operations on the rationals to operations on the reals, and proved that these extensions preserved the usual identities of arithmetic. This aspect of Dedekind's construction has been generalized to other algebraic structures as well (see the references below). Until fairly recently however, the only study in this direction where general classes of algebras are considered, seems to be that of Monk [36], where MacNeille completions of certain Boolean algebras with operators are introduced. The aim of our paper is to investigate, in a systematic manner, MacNeille completions of the much wider class of *lattice expansions*, or *lattice-ordered algebras*.

Definition 1.1. We will call an algebraic similarity type $\mathcal{E}$ a *lattice expansion type* if it does not contain any of the lattice symbols $\wedge$, $\vee$, $\perp$ and $\top$; we let $\mathcal{E}^+$ denote the similarity type obtained by adjoining these lattice symbols to $\mathcal{E}$.

The $\{\vee, \wedge, \perp, \top\}$-reduct, or *lattice reduct* of an algebra $\mathbb{A}$ of type $\mathcal{E}^+$ is denoted as $\mathbb{A}_\flat$. An $\mathcal{E}$-expanded lattice is an $\mathcal{E}^+$-algebra $\mathbb{A}$ such that $\mathbb{A}_\flat$ is, indeed, a lattice. $\lhd$

These structures have been studied in general algebra, but also play a fundamental role in algebraic logic by providing a natural algebraic semantics for virtually any formal logic that allows some kind of reasonable conjunction and disjunction [12]. Examples include Heyting and Boolean algebras[1], ortholattices, residuated lattices, modal algebras, relation algebras, Ockham algebras, and many, many more. As discussed by ONO [39], MacNeille completions of such algebras play an important role in the completeness theory of the corresponding logics.

It turns out that there are two natural ways to extend a map f between two lattices $\mathbb{L}$ and $\mathbb{M}$ to a map between their MacNeille completions $\overline{\mathbb{L}}$ and $\overline{\mathbb{M}}$, leading to the *lower* extension f° and the *upper* extension $f^\bullet$ of f, respectively (precise definitions will be supplied below). Applying this knowledge to the additional operations of a lattice expansion $\mathbb{A}$, we may form various MacNeille completions $\mathbb{A}^\psi$ of $\mathbb{A}$, depending on an extension type ψ which determines for each operation whether to take the upper or the lower extension. In this paper we mainly (but

[1] Since the algebraic structure of Heyting and Boolean algebras is completely determined by their lattice reduct, we can see both of these algebras as special lattices, or as lattice expansions.

not exclusively) restrict our attention to the lower and upper extensions $\mathbb{A}^{\circ}$ and $\mathbb{A}^{\bullet}$ (i.e., the structures we obtain by uniformly taking the lower respectively the upper extension of each additional operation).

As mentioned earlier, the matter of extending additional operations on lattices to operations on their MacNeille completion has been addressed in the literature — we give a brief survey here. We already encountered the examples of Heyting and Boolean algebras as early examples. Possibly the most extensive literature appeared on ortholattices and orthomodular lattices. MacLaren [34] showed that the (unique) MacNeillle completion of an ortholattice is again an ortholattice. This closure property does not apply to the variety of all orthomodular lattices, see Adams [1] or Harding [26], but it is again satisfied by those varieties of orthomodular lattices that are generated by lattices of bounded depth, see Harding [28]. Monk [36] introduced and studied the lower completion of a Boolean algebra with operators. Focusing on complete operators, he proved among other things, that identities not involving the negation symbol are preserved under taking MacNeille completions. MacNeille completions of residuated lattices and similar structures play an important role in the algebraic semantics of substructural logics, and particular, in completeness proofs. This is the main theme in the work of Ono [37, 39, 38], who shows that many classes of residuated lattices are closed under taking MacNeille completions; see also Rosenthal [40] for a connection with quantale theory. Givant & Venema [24] took up the work of Monk on Boolean algebras with operators. Their main result shows that, in case all the primitive operations of the algebra are conjugated (which is on a Boolean basis the same as being residuated), the validity of all Sahlqvist equations is preserved under taking MacNeille completions. Hirsch & Hodkinson [29] study MacNeille completions for various varieties that arise in the algebraic study of relations. Mainly, their results are negative, for instance, they show that the class of representable relation algebras is not MacNeille canonical. Bezhanishvili & Harding [7, 6] have interesting recent work on Heyting algebras and modal algebras. In [7] they prove that only three varieties of Heyting algebras are closed under taking MacNeille completions: the trivial variety, the variety of all Heyting algebras, and that of Boolean algebras. In [6] they characterize the varieties of closure algebras and diagonalizable algebras that are closed under lower and upper completions. Finally, Santocanale [41] studies MacNeille completions of modal algebras that are extended with fixpoint operators.

MacNeille's construction is not the only way to complete a lattice. A particularly interesting alternative to the MacNeille completion is the *canonical extension* which goes back to the work of Jónsson & Tarski [31, 32] on Boolean algebras. In recent years, through the work of Gehrke, Jónsson and others, the notion has been extended to the much wider settings of distributive lattices [20, 21], arbitrary lattices [18], and partial orders [13], respectively. One may define the canonical extension $\mathbb{L}^{\sigma}$ of a lattice $\mathbb{L}$ in a concrete way, (for instance, in the case of a Boolean algebra, as the power set algebra of the dual Stone space), or define it abstractly, in a very similar fashion as the MacNeille completion. Also, any operation on a lattice $\mathbb{L}$ has a lower extension and an upper extension on $\mathbb{L}^{\sigma}$, so that the notion of canonical extension applies to lattice expansions as well — again, parametrized by an extension type.

In the past, the similarities between the two notions have had a substantial influence on the theory of MacNeille completions of Boolean algebras with operators. For instance, MONK [36] builds on JÓNSSON & TARSKI [31], and GIVANT & VENEMA [24] were inspired by JÓNSSON [30]. Likewise, we are indebted to the approach towards lattice expansions of GEHRKE & HARDING [18], and the *topological* methodology of GEHRKE & JÓNSSON [22]. Finally, apart from similarities between the two ways of completing a lattice expansion, there are *connections* as well. GEHRKE, HARDING & VENEMA [19] prove that any canonical extension of a lattice expansion $\mathbb{A}$, can be embedded in the corresponding MacNeille completion of some ultrapower of $\mathbb{A}$ — here 'corresponding' means that the two extension types agree. From this it follows that every variety (in fact, every universal class) that is MacNeille canonical for some extension type, is also closed under taking canonical extensions of the same extension type.

Overview. We first give some basic facts concerning MacNeille completions in section 2. In section 3 we show that there are two natural ways to extend an order preserving lattice map f between two lattices $\mathbb{L}$ and $\mathbb{M}$ to a map between their MacNeille completions $\overline{\mathbb{L}}$ and $\overline{\mathbb{M}}$, leading to the *lower* extension f° and the *upper* extension $f^{\bullet}$ of f, respectively. For a number of properties, we discuss whether a lattice map f with that property is *smooth* (meaning that $f^{\circ} = f^{\bullet}$), and whether the property is preserved under taking lower or upper extensions. We specify the answers to these questions, depending on the nature of the base lattices: a survey of our findings can be found in Table 1.

We unfold the topological perspective on MacNeille completions in section 4. We define three kinds of topologies on (the carrier C of) a complete lattice $\mathbb{C}$ with a fixed sublattice $\mathbb{L}$: two Alexandrov topologies, three Scott topologies, and three topologies $\rho^{\uparrow}$, $\rho^{\downarrow}$ and ρ that are induced by $\mathbb{L}$. We characterize the MacNeille completion of a lattice $\mathbb{L}$ as the unique completion of $\mathbb{L}$ on which ρ is Hausdorff. In the following section we consider the MacNeille extension of lattice maps from this topological perspective. First we generalize the definition of the lower and upper MacNeille extension, from order preserving lattice maps to arbitrary ones. Then we discuss continuity properties (with respect to the earlier given topologies) of the MacNeille extensions of lattice maps. We find interesting and useful continuity properties for the primitive lattice operations and for lattice maps satisfying some kind of residuation property.

In section 6 we turn to preservation results for equational validity; our approach crucially involves the earlier mentioned continuity properties, applied to term functions. Given our earlier findings on the preservation of properties of lattice maps, we focus on the following varieties: lattice expansions with order preserving operations, expansions of Boolean algebras, residuated lattice expansions, and expansions of lattices with operations that either preserve or reverse the order in each of their arguments. In each case we provide reasonably sharp sufficient conditions on the syntactic shape of equations that guarantee preservation.

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2. Lattices and their MacNeille completions

In this paper we presuppose familiarity with the basic concepts and terminology of universal algebra [9], lattice theory [8, 10], and topology [14]. Our notation and terminology will be mostly standard; here we just mention a few specific points in order to avoid confusion in the sequel.

Preliminaries on lattices. All lattices that we consider in this paper will be *bounded*, so that for convenience we define a *lattice* to be an algebra $\mathbb{L} = \langle L, \wedge, \vee, \top, \bot\rangle$ satisfying the familiar laws of bounded lattices. Accordingly, a *lattice homomorphism* is supposed to not only preserve (finite) meets and joins, but also top and bottom. We will generally ignore the distinctions between isomorphic structures in this paper. Meets and joins of arbitrary sets are denoted using the symbols $\bigwedge$ and $\bigvee$, respectively.

Given an element a of a lattice $\mathbb{L}$, the *principal down-set* generated by a is given as ${\downarrow}a := \{u \in L \mid u \leq a\}$. Likewise, the *principal up-set* is defined as ${\uparrow}a := \{u \in L \mid a \leq u\}$. The *interval* determined by two elements $a \leq b$ of $\mathbb{L}$ is the set $[a, b] := \{u \in L \mid a \leq u \leq b\}$.

The *order dual* of a lattice $\mathbb{L} = \langle L, \wedge, \vee, \top, \bot\rangle$, i.e., the structure $\langle L, \vee, \wedge, \bot, \top\rangle$ is denoted as $\mathbb{L}^\partial$. The fact, that $\mathbb{L}^\partial$ is a lattice as well, enables us to shorten quite a lot of definitions and proofs by referring to the *principle of order duality*: Every fact concerning lattices remains valid after swapping $\top$ with $\bot$, $\wedge$ with $\vee$, etc.

On a few occasions we will need to discuss various distributive laws; for instance, we will say that a lattice $\mathbb{L}$ satisfies $(\wedge, \bigvee)$-distributivity if $a \wedge \bigvee B = \bigvee\{a \wedge b \mid b \in B\}$, for all $a \in L$ and $B \subseteq L$.

MacNeille completions of lattices. A lattice $\mathbb{C}$ is a *completion* of a lattice $\mathbb{L}$ if $\mathbb{C}$ is complete and $\mathbb{L}$ is a subalgebra of $\mathbb{C}$. If all existing meets and joins in $\mathbb{L}$ agree with those taken in $\mathbb{C}$, then we call $\mathbb{C}$ a *regular* completion of $\mathbb{L}$, but in general we do not require completions to be regular. Thus the notation $\vee$ for finite joins is unambiguous, but not so for infinite joins. Our convention will be that $\bigvee X$ always denotes $\bigvee^{\mathbb{C}} X$, that is, the join taken in the completion. The same applies to notation like ${\uparrow}a$ or $[a, b]$; in case we want to refer to the elements *inside* L that are below a, or in between a and b, we use the notation $L_{\downarrow x}$ and $L_{[a,b]}$, respectively.

Given a subset A of a complete lattice $\mathbb{C}$, we call an element $c \in C$ *closed (open, respectively) with respect to* A if c is the meet (join, respectively) in $\mathbb{C}$ of elements in A. We let $K_{\mathbb{C}}(A)$ and $O_{\mathbb{C}}(A)$ denote the collections of closed and open elements, respectively. In the sequel, we may write $K_{\mathbb{C}}$, $K(A)$, or even K, instead of $K_{\mathbb{C}}(A)$, if the suppressed details are clear from context; and similarly for the set $O_{\mathbb{C}}(A)$.

We say that a set $A \subseteq C$ is *meet-dense* in $\mathbb{C}$ if $K_{\mathbb{C}}(A) = C$, *join-dense* if $O_{\mathbb{C}}(A) = C$, *doubly dense* if it is both join- and meet-dense, and *dense* if $K_{\mathbb{C}}(O_{\mathbb{C}}(A)) = O_{\mathbb{C}}(K_{\mathbb{C}}(A)) = C$. In words, A is doubly dense in $\mathbb{C}$ if every element of C is both a meet and a join of elements in A. In practice, we will often work with the following characterization of join- and meet-density. We omit the straightforward proof of this proposition.

Proposition 2.1. *Let A be some subset of a complete lattice $\mathbb{C}$. Then the following are equivalent.*

(1) *A is join-dense in $\mathbb{C}$;*
(2) *$x = \bigvee A_{\downarrow x}$ for all $x \in C$;*
(3) *for all x, y in C with $x \not\leq y$ there is an $a \in A$ with $a \leq x$ and $a \not\leq y$.*

For our purposes, the following abstract definition of the MacNeille completion of a lattice is the most convenient.

Definition 2.2. A completion $\mathbb{C}$ of a lattice $\mathbb{L}$ is called a *MacNeille completion of* $\mathbb{L}$ if L is doubly dense in $\mathbb{C}$. $\lhd$

It is in fact a rather strong property for one lattice to be a MacNeille completion of another. To start with, every lattice has a *unique* MacNeille completion.

Theorem 2.3. *Let $\mathbb{L}$ be some lattice. Then*

(1) *(existence) $\mathbb{L}$ has a MacNeille completion;*
(2) *(unicity) any two MacNeille completions of $\mathbb{L}$ are isomorphic via a unique isomorphism that restricts to the identity on L.*

Proof. Fix a lattice $\mathbb{L}$; we first consider *existence*. Given a subset A of L, let A^u be the set of upper bounds of A, and A^l the set of its lower bounds. Then a *cut* is a pair (A, B) of subsets of L such that $A = B^l$ and $B = A^u$. If we order the collection of cuts by putting

$$(A, B) \leq (A', B') \iff A \subseteq A' (\iff B' \subseteq B),$$

we obtain a complete lattice $\mathbb{M}$. This is not hard to prove, using the observation that the cuts of $\mathbb{L}$ can be identified with the stable sets of the Galois connection induced by the order relation of the lattice. It is easy to see that the map $x \mapsto (\downarrow x, \uparrow x)$ is a lattice embedding of $\mathbb{L}$ into $\mathbb{M}$, and that (the image of) $\mathbb{L}$ is in fact doubly dense inside $\mathbb{M}$. The details of these well-known results can be found in for instance [10].

For *unicity*, suppose that L is doubly dense in both $\mathbb{M}$ and $\mathbb{M}'$; define the map $f : M \to M'$ by

$$f(x) := \bigvee{}' L_{\downarrow x}.$$

Again, it is not hard to prove that f is in fact an isomorphism, and that $f(x) = x$ for all $x \in L$. For details, again, we refer to [10]. $\square$

Since we generally identify isomorphic structures in this paper, Theorem 2.3 allows us to speak of *the* MacNeille completion of a lattice.

Definition 2.4. The MacNeille completion of a lattice $\mathbb{L}$ will be denoted as the structure $\overline{\mathbb{L}}$. Abusing notation, we will usually denote the carrier of the MacNeille completion $\overline{\mathbb{L}}$ as $\overline{L}$. $\lhd$

As a further introductory remark, we note that MacNeille completions interact well with products and order duals.

Proposition 2.5. *Let $\mathbb{L}$ a lattice, and $\{\mathbb{L}_i \mid i \in I\}$ a family of lattices. Then*

(1) $\overline{\prod_{i\in I} \mathbb{L}_i} \cong \prod_{i\in I} \overline{\mathbb{L}_i}$,
(2) $\overline{(\mathbb{L}^\partial)} \cong (\overline{\mathbb{L}})^\partial$.

Proof. Both statements can be proved on the basis of Theorem 2.3. For instance, using Proposition 2.1, it is easy to show that the product $\prod_{i\in I} \overline{\mathbb{L}_i}$ is a MacNeille completion of the product $\prod_{i\in I} \mathbb{L}_i$. And using facts like $K(\mathbb{L}^\partial) = O(\mathbb{L})$, the second statement of the proposition immediately follows from the principle of order duality. $\square$

Given our policy of identifying isomorphic structures, the previous proposition justifies our taking the product $\prod_{i\in I} \overline{\mathbb{L}_i}$ of the MacNeille completions to be the MacNeille completion of the product lattice $\prod_{i\in I} \mathbb{L}_i$. In particular, this will enable us to identify any element x of $\overline{\prod_{i\in I} \mathbb{L}_i}$ with a tuple $(x_i)_{i\in I}$ of elements of the individual lattices $\overline{\mathbb{L}_i}$. A similar convention applies to the MacNeille completion of the order dual of a lattice.

We finish the section with some technical details concerning the relation between MacNeille completions and the canonical extensions mentioned in the introduction.

Remark 2.6. In order to define the notion of canonical extension of a lattice, we need the following definition concerning a completion $\mathbb{C}$ of a lattice $\mathbb{L}$. We say that $\mathbb{L}$ is *compact* in $\mathbb{C}$ if for all sets X and Y of closed and open elements, respectively, $\bigwedge X \leq \bigvee Y$ implies the existence of finite sets $X_0 \subseteq X$, $Y_0 \subseteq Y$ such that $\bigwedge X_0 \leq \bigvee Y_0$. Now, analogous to Theorem 2.3 on MacNeille completions, Gehrke & Harding [18] prove that every lattice $\mathbb{L}$ has a unique completion in which $\mathbb{L}$ is both dense and compact. In the case that $\mathbb{L}$ is a Boolean algebra, one may take for $\mathbb{L}^\sigma$ the 'double dual' of $\mathbb{L}$, that is, the power set of the underlying set of the dual Stone space. A similar construction involving Priestley duality works for distributive lattices.

Nevertheless, there are also noteworthy differences between the canonical extension and the MacNeille completion of a lattice. To name one: whereas the MacNeille completion agrees with *all* meets and joins that exist in the original lattice, the canonical extension agrees with *none* of the existing infinite meets or joins. Related to this is the fact that taking canonical extensions commutes with taking *finite* products only: the analogue of the first part of Proposition 2.5 fails for the case that I is infinite.

3. Extensions of order preserving lattice maps

In order to investigate how additional operations on a lattice $\mathbb{L}$ can be extended to operations on the MacNeille completion $\overline{\mathbb{L}}$ of the lattice, it makes sense to slightly broaden the issue to the following question. Given two lattices $\mathbb{L}$ and $\mathbb{M}$, and a map $f : L \to M$, how can we systematically define a map $g : \overline{L} \to \overline{M}$ which extends f, that is, satisfies $g{\restriction}_L = f$? In this section we take up this question, for the time being confining ourselves to *order preserving* maps. It is also convenient to have some terminology.

Definition 3.1. Given two lattices $\mathbb{L}$ and $\mathbb{M}$, a *lattice map* between $\mathbb{L}$ and $\mathbb{M}$ is nothing but a map $f : L \to M$. We will also use the notation $f : \mathbb{L} \to \mathbb{M}$.

A lattice map $f : \mathbb{L} \to \mathbb{M}$ is *isotone* or *order preserving* if for all $x, y \in L$ it holds that $x \leq^{\mathbb{L}} y$ only if $f(x) \leq^{\mathbb{M}} f(y)$, and *order reversing* or *antitone* if $x \leq^{\mathbb{L}} y$ only if $f(x) \geq^{\mathbb{M}} f(y)$. $\lhd$

Note that the *lattice structure* of the domain and codomain lattice is part of the definition of a lattice map. This facilitates the discussion of properties of such maps; for instance, a map $f : L \to M$ is order preserving as a lattice map from $\mathbb{L}$ to $\mathbb{M}$ if and only if it is order reversing as a lattice map from $\mathbb{L}^{\partial}$ to $\mathbb{M}$.

General observations. Given the fact that every element of the MacNeille completion $\overline{\mathbb{L}}$ of a lattice $\mathbb{L}$ is both a join and a meet of elements of $\mathbb{L}$, any isotone map between two lattices $\mathbb{L}$ and $\mathbb{M}$ naturally induces *two* operations between the respective MacNeille completions $\overline{\mathbb{L}}$ and $\overline{\mathbb{M}}$.

Definition 3.2. Let $f : \mathbb{L} \to \mathbb{M}$ be an isotone lattice map. Then we define the lattice maps $f^{\circ}, f^{\bullet} : \overline{\mathbb{L}} \to \overline{\mathbb{M}}$ as follows:

$$\begin{aligned} f^{\circ}(x) &:= \bigvee \{f(a) \mid x \geq a \in L\}, \\ f^{\bullet}(x) &:= \bigwedge \{f(a) \mid x \leq a \in L\}. \end{aligned}$$

These f° and $f^{\bullet}$ are called the *lower* and *upper (MacNeille) extension* of f, respectively. $\lhd$

The following proposition gathers some basic properties of these two operations. We omit its proof, which is completely straightforward.

Proposition 3.3. *Let $f : \mathbb{L} \to \mathbb{M}$ be an isotone lattice map. Then*

(1) *both f° and $f^{\bullet}$ extend f;*
(2) *$f^{\circ} \leq f^{\bullet}$ (that is, $f^{\circ}(x) \leq f^{\bullet}(x)$ for all x in $\overline{\mathbb{L}}$);*
(3) *both f° and $f^{\bullet}$ are isotone lattice maps between $\mathbb{L}$ and $\mathbb{M}$.*

Many natural issues arise concerning these notions. In this section we will address the following questions:

- How do standard lattice operations behave under these constructions?
- Which properties of lattice operations are preserved by the lower/upper extension?
- For which operations do the lower and the upper extension coincide?

Smoothness and canonicity. Starting with the latter question, we first introduce some terminology.

Definition 3.4. An isotone lattice map $f : \mathbb{L} \to \mathbb{M}$ for which $f^\circ = f^\bullet$ is called *smooth*. The extension of a smooth operation f is sometimes denoted as $\overline{f}$. $\lhd$

It goes without saying that smoothness is an interesting and important notion, but, as we will see, it is the exception rather than the rule. Fortunately however, the lattice operations themselves are smooth. In addition, their extensions agree with the lattice operations of the MacNeille completion.

Proposition 3.5. *Let $\mathbb{L}$ be a lattice. Then*

(1) *the join operation $\vee : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ is smooth, and the operation $\vee^\circ = \vee^\bullet$ coincides with the join operation $\overline{\vee} : \overline{\mathbb{L}} \times \overline{\mathbb{L}} \to \overline{\mathbb{L}}$.*
(2) *the meet operation $\wedge : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ is smooth, and the operation $\wedge^\circ = \wedge^\bullet$ coincides with the meet operation $\overline{\wedge} : \overline{\mathbb{L}} \times \overline{\mathbb{L}} \to \overline{\mathbb{L}}$.*

Proof. We only prove the part concerning the join operation, the second statement then follows by order duality. It suffices to prove that $\vee^\bullet \leq \overline{\vee} \leq \vee^\circ$, since we already know that $\vee^\circ \leq \vee^\bullet$ by Proposition 3.3.

For the first inequality, take x_1 and x_2 in $\overline{\mathbb{L}}$, and c in $\mathbb{L}$ such that $x_1 \overline{\vee} x_2 \leq c$. Then both x_1 and x_2 are below c, so $(x_1, x_2) \leq (c, c) \in L^2$. By definition of $\vee^\bullet$, we have

$$x_1 \vee^\bullet x_2 = \bigwedge \{a_1 \vee a_2 \mid (x_1, x_2) \leq (a_1, a_2) \in L^2\},$$

so $x_1 \vee^\bullet x_2 \leq a_1 \vee a_2$ for all a_1, a_2 in $\mathbb{L}$ with $(x_1, x_2) \leq (a_1, a_2)$. So, in particular, we find that $x_1 \vee^\bullet x_2 \leq c \vee c = c$. And since c was an arbitrary element of L, we obtain $x_1 \vee^\bullet x_2 \leq x_1 \overline{\vee} x_2$ by meet-density of $\mathbb{L}$ in $\overline{\mathbb{L}}$ and (the order dual of) Proposition 2.1.

For the second inequality, take elements x_1 and x_2 in $\overline{\mathbb{L}}$, and a c in $\mathbb{L}$ satisfying $x_1 \vee^\circ x_2 \leq c$. Then for all $a_1 \leq x_1$ and $a_2 \leq x_2$ we have $a_1 \vee a_2 \leq c$, whence $x_1 \overline{\vee} x_2 = \bigvee L_{\downarrow x_1} \overline{\vee} \bigvee L_{\downarrow x_2} = \bigvee \{a_1 \vee a_2 \mid a_i \in L_{\downarrow x_i}\} \leq c$. Thus we see, again using meet-density, that $x_1 \overline{\vee} x_2 \leq x_1 \vee^\circ x_2$. $\square$

In the sequel we will see more examples of smooth operations, but we first turn to the second of the above mentioned questions.

Definition 3.6. A property of lattice maps is *lower MacNeille canonical*, or briefly: MacN$^\circ$-*canonical*, if f° has the property whenever f does. The notion of *upper MacNeille canonicity*, briefly: MacN$^\bullet$-*canonicity* is defined analogously. $\lhd$

In the remainder of this section we will discuss which properties of maps are MacNeille canonical, and which ones guarantee smoothness. The properties that we will consider are listed below.

Definition 3.7. Let $f : \mathbb{L} \to \mathbb{M}$ be some lattice map. Then we call f

- *isotone* if it is order preserving,
- *normal* if $f(\bot) = \bot$,
- *additive* or *join preserving* if $f(x \vee y) = f(x) \vee f(y)$, for all $x, y \in L$,
- *multiplicative* or *meet preserving* if $f(x \wedge y) = f(x) \wedge f(y)$, for all $x, y \in L$,

- *a lattice homomorphism* if it is normal, additive, multiplicative and satisfies $f(\top) = \top$,
- *an operator* if $\mathbb{L} = \mathbb{L}_1 \times \cdots \times \mathbb{L}_n$ such that f is additive in each of its arguments,
- *a dual operator* if $\mathbb{L} = \mathbb{L}_1 \times \cdots \times \mathbb{L}_n$ such that f is multiplicative in each of its arguments,
- *completely additive* if $f(\bigvee X) = \bigvee f[X]$ for all $X \subseteq L$ such that $\bigvee X$ exists,
- *completely multiplicative* if $f(\bigwedge X) = \bigwedge f[X]$ for all $X \subseteq L$ such that $\bigvee X$ exists,
- *a complete lattice homomorphism* if it is both completely additive and completely multiplicative,
- *a complete (dual) operator* if $\mathbb{L} = \mathbb{L}_1 \times \cdots \times \mathbb{L}_n$ such that f is completely additive (completely multiplicative, respectively) in each of its arguments,
- *residuated* if it has a *residual*, i.e., a lattice map $g : \mathbb{M} \to \mathbb{L}$ such that for all $x \in L$, $y \in M$ we have $f(x) \leq y$ iff $x \leq g(y)$; such f and g are called a *residual pair*;
- *left residuated* if $\mathbb{L} = \mathbb{L}_1 \times \mathbb{L}_2$ and f has a *left residual*, that is, a lattice map $g : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_1$ such that for all $x_1 \in L_1$, $x_2 \in L_2$ and $y \in M$ we have $f(x_1, x_2) \leq y$ iff $x_1 \leq g(x_2, y)$. The notion of *right residuation* is defined analogously. $\lhd$

We will now see how these properties fare under taking MacNeille completions. As we will see, in some cases better results are possible on the condition that the lattices involved satisfy some additional distributivity laws. As a consequence, maps between Heyting algebras or Boolean algebras in general display better behavior than maps between arbitrary lattices.

Towards the end of this section, we summarize our findings in Table 1.

Isotonicity and normality. There is little of interest to say about the notions of isotonicity and normality: it follows from Proposition 3.3 that isotonicity is both MacN$^\circ$-canonical and MacN$^\bullet$-canonical, while we will provide various counterexamples to smoothness below. Concerning normality, both upper- and lower MacNeille canonicity are trivial to prove; and a counterexample to smoothness can be found in Example 3.8 below.

Additivity, multiplicativity and homomorphisms. The properties of additivity and multiplicativity mark a pronounced difference between MacNeille completions and canonical extensions (as discussed at the end of the previous section). In the latter case, these properties do not only imply smoothness, but one may also show that the canonical extensions of additive (multiplicative) maps are in fact *completely* additive (completely multiplicative), see Gehrke & Harding [18]. For MacNeille completions however, the picture is far less nice. Neither additivity, nor multiplicativity, nor their combination implies smoothness, and furthermore, neither property is preserved under taking either lower or upper MacNeille extensions. Most of these observations even apply in the context of Boolean algebras, as is witnessed by the following example which goes back to Monk [36].

Example 3.8. Let $\mathbb{B}$ be the Boolean algebra of the finite and cofinite sets of natural numbers, and $\mathbf{2}$ the two element Boolean algebra. Now let $f : \mathbb{B} \to \mathbf{2}$ be the lattice map mapping finite sets to 0 and cofinite ones to 1. Then f is both additive and multiplicative — in fact, f is a Boolean homomorphism.

It is easy to see that the MacNeille completion of $\mathbb{B}$ is the full power set algebra of the natural numbers, and that f° and $f^\bullet$ are given by

$$f^\circ(x) = \begin{cases} 1 & \text{if } x \text{ is cofinite,} \\ 0 & \text{otherwise.} \end{cases}$$

$$f^\bullet(x) = \begin{cases} 1 & \text{if } x \text{ is infinite,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly then, f is not smooth, f° is not additive, and $f^\bullet$ is not multiplicative. To see that f° is not additive, let u and v be the sets of all even and all odd numbers, respectively. Then $f^\circ(u \vee v) = f^\circ(\top) = 1$, while $f^\circ(u) \vee f^\circ(v) = 0 \vee 0 = 0$.

In Example 3.8 f° is multiplicative and $f^\bullet$ is additive. This is no coincidence: in case the codomain of a lattice map satisfies some additional distributivity property, multiplicativity becomes MacN$^\circ$-canonical.

Proposition 3.9. *Let $f : \mathbb{L} \to \mathbb{M}$ be some multiplicative lattice map, and assume that $\overline{\mathbb{M}}$ is $(\wedge, \bigvee)$-distributive. Then f° is multiplicative.*

We omit the proof of this proposition, since it it is a special case of Proposition 3.16 below.

As a corollary to Proposition 3.9, we find that multiplicativity is MacN$^\circ$-canonical on the class of Heyting algebras. Note however, that the principle of order duality does not apply to Heyting algebras (that is, the order dual of a Heyting algebra is not necessarily a Heyting algebra). Hence, we may not infer that additivity is MacN$^\bullet$-canonical on the class of Heyting algebras. Boolean algebras on the other hand, form a class of Heyting algebras to which the principle of order duality does apply, so on this class we find not only that multiplicativity is MacN$^\circ$-canonical, but also that additivity is MacN$^\bullet$-canonical.

That the distributivity condition is really needed in Proposition 3.9 can be see from the following example, which shows that on arbitrary lattices, multiplicativity is not MacN$^\circ$-canonical.

Example 3.10. In general, multiplicativity of a lattice map is not preserved under taking lower MacNeille extensions (and hence, by order duality, additivity of a lattice map is generally not MacN$^\bullet$-canonical). Our counterexample is based on a fairly well-known example, due to Funayama [17], showing that the MacNeille completion of a distributive lattice is not necessarily distributive. Without going into the (rather involved) details of this example, we base ourselves on the existence of a lattice $\mathbb{E}$ with the following properties (for details, see the description on p. 238 of [3]).

$\mathbb{E}$ is a distributive lattice, containing two points a_0 and a_1, while the non-distributivity of $\overline{\mathbb{E}}$ is witnessed by three points u_0, u_1 and v such that the five mentioned points form a sublattice of $\overline{\mathbb{E}}$ in the shape of the lattice $\mathbb{N}_5$, see Figure 1.

Now define the operation $\Box : E \to E$ by putting

$$\Box x := a_0 \vee x,$$

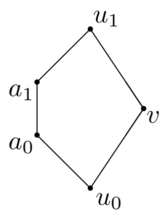

FIGURE 1. the lattice $\mathbb{N}_5$.

then clearly, $\Box$ is multiplicative by the distributivity of $\mathbb{E}$. It is also easy to see that

$$\Box^\circ x = \bigvee\{a_0 \vee b \mid x \geq b \in E\} = a_0 \vee \bigvee\{b \mid x \geq b \in E\} = a_0 \vee x$$

for every x in $\overline{\mathbb{E}}$. But then it immediately follows that $\Box^\circ v \wedge \Box^\circ a_1 = u_1 \wedge a_1 = a_1$, while $\Box^\circ(v \wedge a_1) = \Box^\circ u_0 = a_0$. Thus $\Box^\circ$ is not multiplicative.

Complete additivity, multiplicativity and homomorphisms. The next kind of map that we consider are the lattice maps that preserve *all* (existing) joins, meets, or both. Generally, the results are still negative here: the following example shows that even complete lattice homomorphisms between Heyting algebras need not be smooth, and need not have a completely additive upper MacNeille extension.

Example 3.11. Let $\mathbb{D}$ be the Heyting algebra obtained by putting, on top of the lattice $\mathbb{N}$ of the natural numbers (based on the standard ordering $\leq \omega$), a copy of the order dual $\mathbb{N}^\partial$ of $\mathbb{N}$. Formally, define $\breve{\omega} := \{\breve{n} \mid n \in \omega\}$ be a copy of the natural numbers, and put $\breve{k} \leq_{\breve{\omega}} \breve{n}$ iff $n \leq k$. Now define

$$\begin{array}{rcl} D & := & \omega \cup \breve{\omega} \\ \leq_{\mathbb{D}} & := & \leq_\omega \cup \leq_{\breve{\omega}} \cup (\omega \times \breve{\omega}). \end{array}$$

We leave it as an exercise for the reader to verify that $\overline{\mathbb{D}}$ is obtained from $\mathbb{D}$ by adjoining a single element, to be called ∞, to $\mathbb{D}$, and place it right between the two copies of the natural numbers inside $\mathbb{D}$, as in Figure 2.

Now consider, for some fixed natural number $n > 0$, the map $f_n : D \to D$ given by

$$f_n(x) := \begin{cases} x & \text{if } n > x \in \omega, \\ n & \text{if } n \leq x \in \omega, \\ x & \text{if } x \in \breve{\omega}. \end{cases}$$

It is easy to see that f preserves all existing joins and meets.

Concerning the MacNeille extensions of f_n, one readily checks that $f_n^\circ(\infty) = n$, and $f_n^\bullet(\infty) = \infty$. This clearly witnesses the non–smoothness of f_n.

In addition, f° is not completely multiplicative. Let $A = \breve{\omega}$ be the 'upper part' of $\mathbb{D}$, then $f_n^\circ(\bigwedge A) = f_n^\circ(\infty) = n$, while $\bigwedge f_n^\circ[A] = \bigwedge A = \infty$. Finally, it is equally easy to see that $f_n^\bullet$ is not completely additive: taking $B = \omega$ as the 'lower part' of $\mathbb{D}$, we see that $f_n^\bullet(\bigvee B) = \infty$, but $\bigvee f_n^\bullet[B] = n$.

FIGURE 2. the lattice $\overline{\mathbb{D}}$.

The next example shows that for arbitrary lattices, it is not only the upper extension that fails to preserve complete additivity, the lower extension does so as well.

Example 3.12. Consider the lattices $\mathbb{F}$ and $\overline{\mathbb{F}}$ depicted in Figure 3. $\mathbb{F}$ is the lattice with carrier $A \cup B \cup C$, where $A = \{a_i \mid i \in \omega\}$, etc. It is easy to see that we obtain its completion $\overline{\mathbb{F}}$ by adjoining the single element ∞, as in the figure.

Let $\mathbf{2}$ denote the two element lattice, and consider the lattice map $f : F \to \mathbf{2}$ given by

$$f(x) := \begin{cases} 0 & \text{if } x \in B, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly then, f is completely additive. However, for its lower extension f° we obtain $f^\circ(\infty) = \bigvee\{f(x) \mid x \in F_{\downarrow\infty}\} = 1$, whereas $\bigvee f^\circ[B] = 0$. Since $\infty = \bigvee B$, this clearly shows that f° does not preserve arbitrary joins.

However, in case that the codomain of a lattice map satisfies some infinite distributivity properties, there is good news to report about the *lower* extension.

Proposition 3.13. *Let $f : \mathbb{L} \to \mathbb{M}$ be some completely additive lattice map, and assume that $\overline{\mathbb{L}}$ is $(\wedge, \bigvee)$-distributive. Then f° is completely additive.*

Proof. Let X be an arbitrary subset of $\overline{\mathbb{L}}$. Completely additive maps are certainly isotone, so by Proposition 3.3 it follows that f° is isotone as well. This implies that $\bigvee f^\circ[X] \leq f^\circ(\bigvee X)$; hence, in order to prove the Proposition, it suffices to show the opposite inequality.

First, note that it follows by the join-density of $\mathbb{L}$ inside $\overline{\mathbb{L}}$ that

$$\bigvee X = \bigvee\{\bigvee L_{\downarrow x} \mid x \in X\} = \bigvee\Big(\bigcup_{x\in X} L_{\downarrow x}\Big).$$

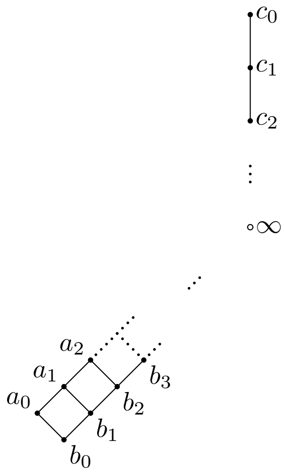

FIGURE 3. the lattice $\overline{\mathbb{F}}$.

Now take an $a \in L$ with $a \leq \bigvee X$. Given the $(\wedge, \bigvee)$-distributivity of $\overline{\mathbb{L}}$, the following holds:

$$a = a \wedge \bigvee X = a \wedge \bigvee \Big(\bigcup_{x \in X} L_{\downarrow x} \Big) = \bigvee \{ a \wedge b \mid b \in \bigcup_{x \in X} L_{\downarrow x} \}.$$

As f is completely additive, this implies that

$$f(a) = \bigvee \{ f(a \wedge b) \mid b \in \bigcup_{x \in X} L_{\downarrow x} \},$$

so that by isotonicity, and the definition of f°, we find

$$f(a) \leq \bigvee \{ f(b) \mid b \in \bigcup_{x \in X} L_{\downarrow x} \} \leq \bigvee f^{\circ}[X].$$

Since the above holds for arbitrary $a \in L$ with $a \leq \bigvee X$, this implies

$$f^{\circ}(\bigvee X) = \bigvee \{ f(a) \mid \bigvee X \geq a \in L \} \leq \bigvee f^{\circ}[X],$$

as required. $\square$

Finally, in the specific case of a complete lattice homomorphism between Boolean algebras, the situation improves completely. The following observation (which was made in a discussion between the second author and N. Bezhanishvili) follows from results by BANASCHEWSKI & BRUNS [5].

Proposition 3.14. *Let $f : \mathbb{L} \to \mathbb{M}$ be a complete lattice homomorphism between the Boolean algebras $\mathbb{L}$ and $\mathbb{M}$. Then f is smooth, and $\overline{f}$ is a complete homomorphism between $\overline{\mathbb{L}}$ and $\overline{\mathbb{M}}$.*

Proof. First observe that it follows from Proposition 3.13 (and its order dual) that f° preserves $\bigvee$ and that $f^{\bullet}$ preserves $\bigwedge$. Also, since f preserves the empty meet and the empty join, it follows that f is normal, and, dually, satisfies $f(\top) = \top$; but then both f° and $f^\bullet$ must satisfy these properties as well. From this we obtain that for all x in $\overline{\mathbb{L}}$:

$$\begin{cases} f^\circ(x) \vee f^\circ(\neg x) &= \top, \\ f^\bullet(x) \wedge f^\bullet(\neg x) &= \bot. \end{cases} \tag{1}$$

From the second identity we may derive that $f^\bullet(x) \leq \neg f^\bullet(\neg x)$, while it follows from Proposition 3.3 that $f^\bullet(\neg x) \geq f^\circ(\neg x)$, so that $\neg f^\bullet(\neg x) \leq \neg f^\circ(\neg x)$. But the first identity of (1) gives that $\neg f^\circ(\neg x) \leq f^\circ(x)$. Putting these three inequalities together, we find that $f^\bullet(x) \leq f^\circ(x)$. This suffices to prove smoothness of f. □

In the above proposition, the map f needs to preserve *both* infinite meets and infinite joins, as the following example witnesses.

Example 3.15. Let $\mathbb{B}$ be the Boolean algebra of the finite and cofinite sets of natural numbers, and let $\mathbf{2}$ be the two element Boolean algebra.

Consider the map $f : \mathbb{B} \to (\mathbf{2} \times \mathbf{2})$ defined as follows:

$$f(a) = \begin{cases} (1,0) & \text{if } a \subseteq e \\ (0,1) & \text{if } a \subseteq \neg e \\ (0,0) & \text{if } a = \varnothing \\ (1,1) & \text{otherwise,} \end{cases}$$

where e and $\neg e$ are the sets of even and odd numbers, respectively. It is fairly easy to show that f preserves all existing joins — we leave the details as an exercise for the reader.

As we saw already, the MacNeille completion of $\mathbb{B}$ is the full power set algebra of the natural numbers, whereas $\mathbf{2} \times \mathbf{2}$, being finite, is its own completion. Concerning f° and $f^\bullet$, it is straightforward to verify that

$$\begin{aligned} f^\circ(e) &= (1,0) \\ f^\bullet(e) &= (1,1), \end{aligned}$$

and that

$$\bigvee\{f^\bullet(a) \mid e \supseteq a \in B\} = \bigvee\{f(a) \mid e \supseteq a \in B\} = (1,0) \neq (1,1) = f^\bullet(e).$$

This clearly shows that f is not smooth, and that $f^\bullet$ is not completely additive.

Operators, dual operators and complete (dual) operators. Since additive (multiplicative) maps are particular instances of operators (dual operators), and likewise for complete (dual) operators, all negative results carry over immediately. Fortunately, however, the positive results that we have found in the Propositions 3.9 and 3.13, can also be obtained for dual operators, and complete operators, respectively.

Proposition 3.16. *Let $f : \mathbb{K} \to \mathbb{M}$ be some lattice map.*

(1) *If f is a dual operator and $\mathbb{M}$ is $(\wedge, \bigvee)$-distributive, then f° is a dual operator.*
(2) *If f is a complete operator and $\mathbb{K}$ is $(\wedge, \bigvee)$-distributive, then f° is a dual operator.*

Proof. For the first part, without loss of generality we may assume that $\mathbb{K}$ is of the form $\mathbb{L} \times \mathbb{L}'$ such that f is multiplicative in both arguments. We will show that f° is multiplicative in its first argument. Take arbitrary elements x, y in $\overline{\mathbb{L}}$, z' in $\overline{\mathbb{L}'}$. Then

$$\begin{aligned}
f^\circ(x,z') \wedge f^\circ(y,z') &= \bigvee\{f(a,c') \mid a \in L_{\downarrow x}, c' \in L'_{\downarrow z'}\} \wedge \bigvee\{f(b,d') \mid b \in L_{\downarrow y}, d' \in L'_{\downarrow z'}\} \\
&= \bigvee\{f(a,c') \wedge f(b,d') \mid a \in L_{\downarrow x}, b \in L_{\downarrow y}, c', d' \in L'_{\downarrow z'}\} \\
&\leq \bigvee\{f(a,c' \vee d') \wedge f(b,c' \vee d') \mid a \in L_{\downarrow x}, b \in L_{\downarrow y}, c', d' \in L'_{\downarrow z'}\} \\
&= \bigvee\{f(a,e') \wedge f(b,e') \mid a \in L_{\downarrow x}, b \in L_{\downarrow y}, e' \in L'_{\downarrow z'}\} \\
&= \bigvee\{f(a \wedge b,e') \mid a \in L_{\downarrow x}, b \in L_{\downarrow y}, e' \in L'_{\downarrow z'}\} \\
&= \bigvee\{f(c,e') \mid c \in L_{\downarrow x \wedge y}, e' \in L'_{\downarrow z'}\} \\
&= f^\circ(x \wedge y, z')
\end{aligned}$$

Since it follows from the isotonicity of f° that $f^\circ(x \wedge y, z') \leq f^\circ(x,z') \wedge f^\circ(y,z')$, this suffices to prove the first part of the proposition.

For the second part, suppose that f is a complete operator. Without loss of generality we may assume that $\mathbb{K}$ is of the form $\mathbb{L} \times \mathbb{L}'$ such that f is completely additive in both arguments. We will show that f° is completely additive in its first argument.

We will show that f° is completely additive in its first argument. Take a subset $X \subseteq \overline{L}$ and an element x' in $\overline{\mathbb{L}'}$. Similar to the proof of Proposition 3.13, it suffices to show that $f^\circ(\bigvee X, x') \leq \bigvee f^\circ[X, x']$, where $f^\circ[X, x']$ is shorthand for the set $\{f^\circ(x,x') \mid x \in X\}$. Now let $a \in L$ and $a' \in L'$ be arbitrary elements such that $a \leq \bigvee X$ and $a' \leq x'$. Following the argumentation of the proof of Proposition 3.13, but for the map $a \mapsto f(a,a')$ instead of for f, we find that $f(a,a') \leq \bigvee f^\circ[X,x']$. From this it follows that

$$f^\circ(\bigvee X, x') = \bigvee\{f(a,a') \mid \bigvee X \geq a \in L, x' \geq a' \in L'\} \leq \bigvee f^\circ[X,x'],$$

as required. □

The above proposition has various kinds of consequences for maps between Heyting algebras or between Boolean algebras, but rather than mentioning these results explicitly here, we refer instead to Table 1.

Residuation. To finish of this section, we discuss lattice maps that enjoy some kind of residuation property. As we will now see, residuation guarantees good behavior under taking MacNeille extensions. To start with, residuation is always lower MacNeille canonical; as far as we know, this result stems from Ono [37]. For a proof, consider the case of a map $f : \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{M}$ that has a left residual the lattice map $g : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_1$. Note that since g is generally not isotone, its MacNeille extensions have not yet been defined. (MacNeille extensions of arbitrary maps will be given in section 5.) Here we simply remark that if f is isotone, then

g is antitone in its first coordinate, and isotone in its second. As we will see in section 5, in such a case, we may put:

$$g^\bullet(x, y) := \bigwedge\{g(a, b) \mid x \leq a \in L_2, y \geq b \in M\}. \tag{2}$$

Proposition 3.17. *Let $f : \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{M}$ be an order preserving map which is left residuated by the lattice map $g : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_1$. Then $f^\circ : \overline{\mathbb{L}_1} \times \overline{\mathbb{L}_2} \to \overline{\mathbb{M}}$ is left residuated by the lattice map $g^\bullet : \overline{\mathbb{L}_2} \times \overline{\mathbb{M}} \to \overline{\mathbb{L}_1}$.*

Proof. Let f and g be as stated above. Then for all $x_1 \in L_1$, $x_2 \in L_2$ and $y \in M$ we have the following chain of equivalences:

$$\begin{aligned}
f^\circ(x_1, x_2) \leq y \quad &\iff \quad f(a_1, a_2) \leq y \text{ for all } a_1 \in (L_1)_{\downarrow x_1}, a_2 \in (L_2)_{\downarrow x_2}, \\
&\iff \quad f(a_1, a_2) \leq b \text{ for all } a_1 \in (L_1)_{\downarrow x_1}, a_2 \in (L_2)_{\downarrow x_2}, b \in M_{\uparrow y}, \\
&\iff \quad a_1 \leq g(a_2, b) \text{ for all } a_1 \in (L_1)_{\downarrow x_1}, a_2 \in (L_2)_{\downarrow x_2}, b \in M_{\uparrow y}, \\
&\iff \quad x_1 \leq g(a_2, b) \text{ for all } a_2 \in (L_2)_{\downarrow x_2}, b \in M_{\uparrow y}, \\
&\iff \quad x_1 \leq g^\bullet(x_2, y),
\end{aligned}$$

which proves the proposition. □

Remark 3.18. As an immediate corollary of Proposition 3.17, it follows that if $\mathbb{H}$ is a Heyting algebra, then so is its upper MacNeille completion $\mathbb{H}^\bullet$, which is defined as the expansion of the MacNeille completion $\overline{\mathbb{H}_\flat}$ of the lattice reduct $\mathbb{H}_\flat$, with the upper extension $\to^\bullet$ of the Heyting implication $\to$ of $\mathbb{H}$.

In the case of a residuated *unary* map, we can even prove smoothness. Recall that residuated unary maps are order preserving, and so are their residuals.

Proposition 3.19. *Let $f : \mathbb{L} \to \mathbb{M}$ and $g : \mathbb{M} \to \mathbb{L}$ form a residual pair. Then both f and g are smooth, and their extensions $\overline{f}$ and $\overline{g}$ also form a residual pair.*

Proof. The proof that f° and $g^\bullet$ form a residual pair is completely analogous to that of the previous proposition. In order to prove that f is smooth, it suffices to show that $f^\bullet \leq f^\circ$. To that aim, take an arbitrary element x of $\overline{\mathbb{L}}$, an arbitrary c in $\mathbb{L}$, and suppose that $f^\circ(x) \leq c$. This means that for all $a \in L_{\downarrow x}$ we have $f(a) \leq c$, so $a \leq g(c)$. From this it follows that x, being the supremum of $L_{\downarrow x}$, is below $g(c)$. But then by isotonicity of $f^\bullet$ we find that $f^\bullet(x) \leq f^\bullet(g(c)) = f(g(c)) \leq c$, where the latter inequality follows from $g(c) \leq g(c)$, by residuation. Since c was an arbitrary element of $\mathbb{L}$, it follows from the meet-density of $\mathbb{L}$ in $\overline{\mathbb{L}}$ that $f^\bullet(x) \leq f^\circ(x)$. □

The result concerning smoothness cannot be generalized to maps that are residuated coordinatewise, as the following example, which is due to Givant & Venema [24], shows.

Example 3.20. It is easy to see that a binary operation f on a Boolean algebra $\mathbb{B}$ is both left- and right-residuated in case it is *self-conjugated*, that is, satisfies

$$f(x, y) \wedge z = \bot \iff f(z, y) \wedge x = \bot \iff f(x, z) \wedge y = \bot.$$

We will now briefly recount the example, due to Givant and the second author, of a non-smooth, self-conjugated operation on a Boolean algebra.

Let S be the set of ordinals less than or equal to ω, and let $\mathbb{B}$ be the field of subsets of S that are either finite or cofinite. Now let R be the ternary relation on the set S that consists of all triples (n, n, n) with $n < \omega$, together with all permutations of the triples $(2n, 2n + 1, \omega)$ with $n < \omega$. We leave it for the reader to verify that the operation f_R, given by

$$f_R(a_1, a_2) := \{n_0 \in S \mid (n_0, n_1, n_2) \in R \text{ for some } n_1 \in a_1, n_2 \in a_2\}, \tag{3}$$

provides a self-conjugated operation on $\mathbb{B}$. It is easy to see that the lower MacNeille extension of f_R is the map $f_R^\circ : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$ defined by (3). In particular, one may calculate that $f_R^\circ(e, e) = e$, where e is the set of even natural numbers. On the other hand, for the upper extension $f_R^\bullet$ of f_R, it can be proven that $f_R^\bullet(e, e) = e \cup \{\omega\}$. This proves that f_R is not smooth, and that $f_R^\circ(e, e)$ is not left-residuated; in fact, it is not even a complete operator.

To finish this discussion of residuation and MacNeille completions, let us mention some examples.

Example 3.21. As a first example, we use Proposition 3.19 to show that every *projection function* $\pi_i : \mathbb{L}_1 \times \cdots \mathbb{L}_n \to \mathbb{L}_i$ is smooth. In order to simplify notation we assume that $n = 2$ and $i = 1$.

It is straightforward to verify that the lattice map $x \mapsto (x, \top)$ between $\mathbb{L}_1$ and $\mathbb{L}_1 \times \mathbb{L}_2$ is the residual of π_1, and that π_1 itself is the residual of the map $x \mapsto (x, \bot) : \mathbb{L}_1 \to \mathbb{L}_1 \times \mathbb{L}_2$. Thus π_i is both a residual and residuated, and hence, smoothness follows from Proposition 3.19. The fact that $\overline{\pi_i}$ coincides with the i-th projection function from $\overline{\mathbb{L}}_1 \times \cdots \overline{\mathbb{L}}_n$ onto $\overline{\mathbb{L}}_i$ is immediate from the definitions and Proposition 2.5.

Example 3.22. Our second and last example concerns the *global discriminator* $\blacklozenge$. Given a lattice $\mathbb{L}$, define the function $\blacklozenge_{\mathbb{L}} : L \to L$ to be the map

$$\blacklozenge_{\mathbb{L}} x := \begin{cases} \bot & \text{if } x = \bot, \\ \top & \text{otherwise.} \end{cases}$$

The name 'global discriminator' stems from the context of Boolean algebras with operators, where the existence of such an operator is equivalent to the existence of a discriminator term in the ordinary universal algebraic sense of the word, see [43] for details.

It is in fact easy to see that $\blacklozenge_{\mathbb{L}}$ is residuated by the map $\blacksquare_{\mathbb{L}} : L \to L$ given by

$$\blacksquare_{\mathbb{L}} x := \begin{cases} \bot & \text{if } x < \top, \\ \top & \text{otherwise.} \end{cases}$$

It then follows that both $\blacklozenge_{\mathbb{L}}$ and $\blacksquare_{\mathbb{L}}$ are smooth, and that $\blacklozenge_{\overline{\mathbb{L}}}$ is residuated by $\blacksquare_{\overline{\mathbb{L}}}$. In fact, one can also show that the MacNeille extension of $\blacklozenge_{\mathbb{L}}$ is the global discriminator of $\overline{\mathbb{L}}$.

Summary. We summarize our findings in Table 1, which lists the properties discussed in this section, indicating (i) how the property fares under taking lower and upper extensions, respectively, and (ii) whether the property guarantees smoothness of the operation. Each of these questions is answered separately for the case that the domain and target are arbitrary lattices, Heyting algebras, or Boolean algebras. Possible answers are '+' (positive), '−' (negative) and '?' (unknown).

Roughly speaking, what the table seems to indicate is that positive results concerning MacNeille extensions of lattice maps are possible in the following two cases:

(1) the lattices satisfy some infinite distributive laws, and/or
(2) the lattice map satisfies some residuation properties.

In order to facilitate the verification of the entries provided by the table, all definite entries ('+' and '−') are accompanied by an explicit justification, in the form of the number of the corresponding item (either counterexample or proposition) in the text. Note that this justification may be somewhat indirect in that the mentioned item may refer to the *order dual* of the corresponding property.

For instance, consider the entries '$-_{3.8}$' for the upper extension of a dual operator. Example 3.8 is not about dual operators directly. However, it does show that additivity of an operation is not preserved under taking lower extension, even in the underlying lattices are Boolean algebras. In fact, the example shows that the operation f° is not an operator either, and so by order duality the upper extension of a double operator is not necessarily a double operator. If this does not hold for a double operator between Boolean algebras then a fortiori the answer is negative for Heyting algebras and arbitrary lattices.

Also, observe that in many cases of a negative answer, the text of this section in fact provides more than one counterexample. For instance, the fact that being a complete operator is not preserved under upper MacNeille extensions, is not only witnessed by Example 3.20 as indicated by the entry in the Table, but also by Example 3.11.

Finally, the question marks indicate open problems. (Note that other than for lattices and Boolean algebras, we may not apply the the principle of order duality here, since the order dual of a Heyting algebra need to be a Heyting algebra.) In each of these cases we expect the answers to be negative, but were not able to find examples witnessing this conjecture.

4. Topologies

In this section we introduce the topological perspective on the MacNeille completion, following the approach taken by Gehrke & Jónsson [22] towards the *canonical extension* of a (distributive) lattice. As we will see in section 6, an important application of the topological approach lies in the study of *MacNeille-canonical* equations, that is, equations that remain valid when we move from a lattice expansion to its MacNeille completion.

We will consider no less than eight topologies on the carrier set $\overline{L}$ of the MacNeille completion of a lattice $\mathbb{L}$. This may seem to be somewhat overwhelming at first, but fortunately, these topologies can be neatly divided in three families, two of which consist of an upper, a lower, and a join topology, and one simply of an upper and a lower topology. Furthermore, two of these families consist of quite well-known

	lower extension			upper extension			smoothness		
	L	HA	BA	L	HA	BA	L	HA	BA
isotonicity	+ 3.3.3	+ 3.3.3.	+ 3.3.3	+ 3.3.3	+ 3.3.3.	+ 3.3.3	− 3.8	− 3.8	− 3.8
normality	+ 3.3.1	+ 3.3.1	+ 3.3.1	+ 3.3.1	+ 3.3.1	+ 3.3.1	− 3.8	− 3.8	− 3.8
additivity	− 3.8	− 3.8	− 3.8	− 3.10	?	+ 3.9	− 3.8	− 3.8	− 3.8
multiplicativity	− 3.10	+ 3.9	+ 3.9	− 3.8	− 3.8	− 3.8	− 3.8	− 3.8	− 3.8
latt homomorph	− 3.8	− 3.8	− 3.8	− 3.8	− 3.8	− 3.8	− 3.8	− 3.8	− 3.8
cpl additivity	− 3.12	+ 3.13	+ 3.13	− 3.11	− 3.11	− 3.15	− 3.11	− 3.11	− 3.15
cpl multiplicativity	− 3.11	− 3.11	− 3.15	− 3.12	?	+ 3.13	− 3.11	− 3.11	− 3.15
cpl latt homomorph	− 3.11	− 3.11	+ 3.14	− 3.11	?	+ 3.14	− 3.11	− 3.11	+ 3.14
operator	− 3.8	− 3.8	− 3.8	− 3.10	?	+ 3.16	− 3.8	− 3.8	− 3.8
dual operator	− 3.11	+ 3.16	+ 3.16	− 3.8	− 3.8	− 3.8	− 3.8	− 3.8	− 3.8
cpl operator	− 3.11	+ 3.16	+ 3.16	− 3.20	− 3.20	− 3.20	− 3.20	− 3.20	− 3.20
cpl dual operator	− 3.20	− 3.20	− 3.20	− 3.11	?	+ 3.16	− 3.20	− 3.20	− 3.20
residuated	+ 3.19	+ 3.19	+ 3.19	+ 3.19	+ 3.19	+ 3.19	+ 3.19	+ 3.19	+ 3.19
left residuated	+ 3.17	+ 3.17	+ 3.17	− 3.20	− 3.20	− 3.20	− 3.20	− 3.20	− 3.20

TABLE 1. Properties of MacNeille extensions of lattice maps

topologies, and finally, there are many connections between the topologies from different families.

In any case, the reason for introducing these topologies is that many interesting properties of lattice maps can be formulated as continuity properties with respect to some of these topologies. As a terminological convention, let us call a map between the sets A and B (τ, τ')-*continuous*, if it is a continuous function between the topological spaces $\langle A, \tau \rangle$ and $\langle B, \tau' \rangle$.

We start with the definition of the two well-known families of topologies; both of them can in fact be defined on arbitrary partial orders.

Definition 4.1. Fix a partial order $\mathbb{P} = (P, \leq)$. The *Alexandrov* topology $\alpha^{\uparrow}$ on P is defined as the collection of all up-sets of $\mathbb{P}$, and the *co-Alexandrov* topology $\alpha^{\downarrow}$ is given, using the principle of order duality, as the collection of all down-sets.

A subset U of $\mathbb{P}$ is called *Scott open* if U is an up-set such that $U \cap D \neq \varnothing$ for every up-directed set D with $\bigvee D \in U$. The *Scott topology* is defined as the collection $\gamma^{\uparrow}$ of Scott open sets; the *co-Scott* topology $\gamma^{\downarrow}$ is defined through the principle of order duality, and we define γ as the join of $\gamma^{\uparrow}$ and $\gamma^{\downarrow}$ (in the lattice of topologies on P). That is, as a basis for γ we may take the set $\{U \cap V \mid U \in \gamma^{\uparrow}, V \in \gamma^{\downarrow}\}$. $\lhd$

Both the Alexandrov and the Scott topologies are defined purely in terms of the order relation, and it should come as no surprise that some familiar order-related properties of lattice maps can be expressed very naturally as continuity properties with respect to these topologies. For instance, it is easy to see that any map between partial orders is isotone iff it is $(\alpha^{\uparrow}, \alpha^{\uparrow})$-continuous iff it is $(\alpha^{\downarrow}, \alpha^{\downarrow})$-continuous. As a second (well-known) example, a map between partial orders is *Scott continuous* (that is, $(\gamma^{\uparrow}, \gamma^{\uparrow})$-continuous) iff it preserves up-directed joins. This can be easily derived using the characterization of the Scott *closed* sets as the down-sets that are closed under taking up-directed joins. In the sequel, the two just mentioned facts will be used without further notice.

We now turn to the second family of topologies. It is easy to see that for any completion $\mathbb{C}$ of a lattice $\mathbb{L}$, the following collections of subsets of C are closed under taking finite intersections: $\{\uparrow a \subseteq C \mid a \in L\}$, $\{\downarrow a \subseteq C \mid a \in L\}$ and $\{[a, b] \subseteq C \mid a, b \in L\}$. It follows that these sets may be taken as *bases* for topologies.

Definition 4.2. Let $\mathbb{C}$ be a completion of the lattice $\mathbb{L}$. The topology on C that is generated by the basis $\{\uparrow a \subseteq C \mid a \in L\}$ is denoted as $\rho^{\uparrow}_{L}(\mathbb{C})$; dually, $\rho^{\downarrow}_{L}(\mathbb{C})$ is generated by the basis $\{\downarrow a \subseteq C \mid a \in L\}$. Finally, we define $\rho_{L}(\mathbb{C})$ as the topology on C that is obtained from taking the set $\{[a, b] \subseteq C \mid a, b \in L\}$ as a basis.

In case $\mathbb{C}$ is the completion of $\mathbb{L}$, we will denote these topologies as $\rho^{\uparrow}_{L}$, $\rho^{\downarrow}_{L}$ and ρ_{L}, respectively. We may also write $\rho^{\uparrow}$, $\rho^{\downarrow}$ and ρ, if the lattice $\mathbb{L}$ is clear from context. $\lhd$

Remark 4.3. These ρ topologies are a variation of the σ topologies ($\sigma^{\uparrow}$, $\sigma^{\downarrow}$ and σ) introduced by Gehrke & Jónsson [22] on the *canonical extension* $\mathbb{L}^{\sigma}$ of a lattice $\mathbb{L}$. For instance, where $\rho^{\uparrow}$ on $\overline{\mathbb{L}}$ is generated by the sets of the form $\uparrow a$ with a a *clopen* element of $\overline{\mathbb{L}}$ (that is, $a \in L$), $\sigma^{\uparrow}$ is generated by sets of the form $\uparrow p$ with p a *closed* element, that is, $p \in K_{\mathbb{L}^{\sigma}}(L)$.

As we will see in the next section, the value of the ρ-topological family lies in the theory of the extension of maps between lattices to maps between their MacNeille completions. The next theorem is the key in this application.

Theorem 4.4. *Let $\mathbb{L}$ be some lattice, and $\overline{\mathbb{L}}$ its MacNeille completion. Then*

(1) *$(\overline{L}, \rho_{L})$ is a zerodimensional Hausdorff space;*
(2) *L is a (topologically) dense subset of $(\overline{L}, \rho_{L})$;*
(3) *L provides the collection of isolated points of ρ_{L}.*

Proof. For the first part, take two distinct points x and y in $\overline{\mathbb{L}}$, then without loss of generality we may assume that $x \not\leq y$. Hence by Proposition 2.1 there is an element a in $\mathbb{L}$ such that $x \not\leq a$, while $y \leq a$. Using the same proposition once more, we find a $b \in L$ with $b \leq x$, $b \not\leq a$. Then $\downarrow a$ and $\uparrow b$ are disjoint open sets separating x and y, which proves that $(\overline{L}, \rho_L)$ is Hausdorff.

It also follows from the above reasoning that

$$\overline{L} \setminus \downarrow a \;=\; \bigcup \{\uparrow b \mid a \not\geq b \in L\},$$

which shows that $\overline{L} \setminus \downarrow a$ is ρ-open, and hence, ρ-*clopen*. From this it is almost immediate that the set $\{[a, b] \subseteq \overline{L} \mid a, b \in L\}$ is in fact a clopen basis for ρ. Thus ρ is zerodimensional indeed.

Part 2 and 3 of the Theorem are easy consequences of the definitions. For instance, density of L is immediate from the fact that ρ_L is generated from basic opens of the form $[a, b]$ with a and b in L. It follows from precisely the same fact that every singleton $\{a\} = [a, a]$ is open, if a belongs to $\mathbb{L}$; this shows that every element of L is an isolated point of ρ_L. □

In passing we note that the ρ topology provides a nice way to *characterize* the MacNeille completion of a lattice (or in fact, of a partial order).

Theorem 4.5. *Let $\mathbb{L}$ be some lattice, and $\mathbb{C}$ some completion of $\mathbb{L}$. Then $\mathbb{C}$ is the MacNeille completion of $\mathbb{L}$ iff $(C, \rho_L^{\mathbb{C}})$ is a Hausdorff space.*

Proof. The direction from left to right has been proven in the previous theorem. For the other direction, let x and y be arbitrary elements of C such that $x \neq y$. Without loss of generality we may assume that $x \not\leq y$, so that $x \wedge y < x$.

Now since $\rho_L(\mathbb{C})$ is Hausdorff, there are disjoint basic $\rho_L(\mathbb{C})$-open sets U_{xy} and U_x containing $x \wedge y$ and x, respectively. Then by definition, there are points a_{xy}, b_{xy}, a_x and b_x, all in $\mathbb{L}$, such that $U_{xy} = [a_{xy}, b_{xy}]$ and $U_x = [a_x, b_x]$. Now suppose for contradiction that $a_x \leq x \wedge y$; then we find that $a_x \leq x \wedge y \leq x \leq b_x$, implying that $x \wedge y \in [a_x, b_x]$ and thus contradicting the disjointness of U_{xy} and U_x.

Hence we see that $a_x \leq x$ but $a_x \not\leq x \wedge y$, from which it immediately follows that $a_x \not\leq y$. Since $a_x \in L$, this establishes the join-density of $\mathbb{L}$ in $\mathbb{C}$ by Proposition 2.1. Meet-density then follows by order duality, and so we have proved that $\mathbb{L}$ is doubly dense in $\mathbb{C}$, i.e., $\mathbb{C}$ is the MacNeille completion of $\mathbb{L}$. □

To finish this section, we gather some useful basic facts connecting the three topological families.

Proposition 4.6. *Let $\overline{\mathbb{L}}$ be the MacNeille completion of some lattice $\mathbb{L}$. Then*

(1) $\gamma^{\uparrow} = \gamma \cap \alpha^{\uparrow}$ *and* $\gamma^{\downarrow} = \gamma \cap \alpha^{\downarrow}$,
(2) $\rho^{\uparrow} = \rho \cap \alpha^{\uparrow}$ *and* $\rho^{\downarrow} = \rho \cap \alpha^{\downarrow}$,
(3) $\gamma^{\uparrow} \subseteq \rho^{\uparrow}$, $\gamma^{\downarrow} \subseteq \rho^{\downarrow}$ *and* $\gamma \subseteq \rho$.

Proof. The proofs of the first two items are trivial and hence, omitted. For the last part, we only prove that $\gamma^{\uparrow} \subseteq \rho^{\uparrow}$; the second statement is then immediate by order duality, and the last statement is easily seen to follow from the first two.

Let U be some $\gamma^{\uparrow}$-open set, and take an arbitrary element $x \in U$. As the set $L_{\downarrow x} = \{a \in L \mid a \leq x\}$ is an up-directed set with $\bigvee L_{\downarrow x} = x \in U$, it follows from $U \in \gamma^{\uparrow}$ that $L_{\downarrow x} \cap U \neq \varnothing$. That is, there must be some $a \in L \cap U$ with $a \leq x$. In other words, we have $x \in {\uparrow}a \in \rho^{\uparrow}$, but also ${\uparrow}a \subseteq U$ since $a \in U$ and U is upwards closed. Since x was an arbitrary element of U, this gives $U = \bigcup\{{\uparrow}a \mid a \in U \cap L\}$, so clearly $U \in \rho^{\uparrow}$. □

5. Topological properties of extended operations

Following the approach of Gehrke & Jónsson [22] towards the canonical extension of lattice expansions, in this section we develop the topological perspective on the extension of lattice maps to maps between the MacNeille completions.

Topological definition of map extensions. In (2) we already encountered a MacNeille extension of a lattice map that was not order preserving. Now we will see how to extend an *arbitrary* map between two lattices $\mathbb{L}$ and $\mathbb{M}$ to maps between the MacNeille completions $\overline{\mathbb{L}}$ and $\overline{\mathbb{M}}$. We base ourselves on the following definition from Gehrke & Jónsson [22].

Definition 5.1. Let X be a dense set in a topological space $\mathbb{S} = \langle S, \tau\rangle$, and let $f : X \to C$ be a map from X to the carrier C of a complete lattice $\mathbb{C}$. Then

$$\begin{array}{rcl} \underline{\lim}_{\tau} f(x) & := & \bigvee\{\bigwedge f[U \cap X] \mid x \in U \in \tau\}, \\ \overline{\lim}_{\tau} f(x) & := & \bigwedge\{\bigvee f[U \cap X] \mid x \in U \in \tau\}. \end{array} \tag{4}$$

are maps from S to C. ◁

We first list some properties of these maps. For a proof of the first proposition, the reader is referred to Gehrke & Jónsson [22].

Proposition 5.2. *Let $\mathbb{S} = \langle S, \tau\rangle$ be some topological space, and let X be some (topologically) dense subset of S. Then*

(1) *restricted to X, $\underline{\lim}_{\tau} f \leq f \leq \overline{\lim}_{\tau} f$, with identity holding on isolated points;*
(2) $\underline{\lim}_{\tau} f \leq \overline{\lim}_{\tau} f$*;*
(3) $\underline{\lim}_{\tau} f$ *is $(\tau, \gamma^{\uparrow})$-continuous, and* $\overline{\lim}_{\tau} f$ *is $(\tau, \gamma^{\downarrow})$-continuous.*

The following Proposition states that the maps defined above really provide a generalization of the upper and lower MacNeille extensions of a isotone map. For its proof, recall from Theorem 4.4 that for any lattice $\mathbb{L}$, its carrier L forms a dense set of the topological space $\langle \overline{L}, \rho\rangle$, and consists entirely of isolated points.

Proposition 5.3. *Let f be an isotone lattice map between $\mathbb{L}$ and $\mathbb{M}$. Then $f^{\circ} = \underline{\lim}_{\rho} f$ and $f^{\bullet} = \overline{\lim}_{\rho} f$.*

Proof. It is immediate to prove, from the fact that the set $\{[a, b] \subseteq \overline{L} \mid a, b \in L \text{ with } a \leq b\}$ is a basis for ρ_L, that

$$\underline{\lim}_{\rho} f(x) \;=\; \bigvee\{\bigwedge f[L_{[a,b]}] \mid L \ni a \leq x \leq b \in L\}.$$

But if f is isotone, then it holds for all $a, b \in L$ that

$$\bigwedge f[L_{[a,b]}] = f(a),$$

so that we find

$$\underline{\lim}_\rho f(x) = \bigvee\{f(a) \mid x \geq a \in L\},$$

from which it easily follows that $\underline{\lim}_\rho f(x) = f^\circ(x)$. The proof for $\overline{\lim}_\rho f(x)$ follows by order duality. □

These observations justify the following definition.

Definition 5.4. Let $f : \mathbb{L} \to \mathbb{M}$ be a lattice map. Then the *lower MacNeille extension* of f is defined as the map $f^\circ : \overline{\mathbb{L}} \to \overline{\mathbb{M}}$ given as $f^\circ = \underline{\lim}_\rho f$, and the *upper MacNeille extension* of f is likewise defined as $f^\bullet = \overline{\lim}_\rho f$. ◁

Remark 5.5. The above definition is based on that of the two *canonical* extensions of a lattice map, from Gehrke & Jónsson [22]. Their idea was to define, for a lattice map $f : \mathbb{L} \to \mathbb{M}$, two lattice maps $f^\sigma, f^\pi : \mathbb{L}^\sigma \to \mathbb{M}^\sigma$, by putting $f^\sigma := \underline{\lim}_\sigma f$ and $f^\pi := \overline{\lim}_\sigma f$.

We first gather some basic properties of these extensions. The following is immediate by the Propositions 5.2 and 5.3.

Proposition 5.6. *Let f be a lattice map between $\mathbb{L}$ and $\mathbb{M}$. Then*

(1) *both f° and $f^\bullet$ extend f, and $f^\circ \leq f^\bullet$,*
(2) *f° is $(\rho, \gamma^\uparrow)$-continuous and $f^\bullet$ is $(\rho, \gamma^\downarrow)$-continuous.*

Uniform maps. Most of the properties that hold for isotone maps can in fact be generalized to the wider class of *uniform* maps. These are the ones that can be presented as a map from some lattice $\mathbb{L}_0 \times \cdots \mathbb{L}_{n-1}$ to a lattice $\mathbb{M}$ which is order preserving in some, and order reversing in the other coordinates.

Definition 5.7. An element $\mu \in \{1, \partial\}^n$ will be called an *order type* or *monotonicity type*. Given such an order type $\mu = (\mu_0, \ldots, \mu_{n-1})$, a map $f : \mathbb{L} \to \mathbb{M}$ is *μ-monotone* if there is a finite direct decomposition $\mathbb{L} \cong \mathbb{L}_0 \times \cdots \mathbb{L}_{n-1}$ such that f is isotone when seen as a lattice map between $\mathbb{L}_0^{\mu_0} \times \ldots \times \mathbb{L}_{n-1}^{\mu_{n-1}}$ and $\mathbb{M}$. $f : \mathbb{L} \to \mathbb{M}$ is *uniform* if it is μ-monotone for some monotonicity type μ. ◁

Example 5.8. Heyting implications and other left residuals maps are order reversing in their first argument, and order preserving in their second argument. In the terminology of Definition 5.7 then, they are $(\partial, 1)$-monotone. Order reversing maps are simply ∂-monotone.

One of the properties of order preserving maps that extend to uniform maps, is that the MacNeille extensions have a simpler definition than in the general case. For instance, if $f : \mathbb{L}_0 \times \ldots \times \mathbb{L}_{n-1} \to M$ is μ-monotone, then we have:

$$f^\circ(x_0, \ldots, x_{n-1}) \;=\; \bigvee\{f(a_0, \ldots, a_{n-1}) \mid x_i \geq^{\mu_i} a_i \in L\},$$

where $\geq^1$ and $\geq^\partial$ denote $\geq$ and $\leq$, respectively. Recall that in (2) on page 17 we already saw an example of this simpler definition, for the upper extension.

Even more important is the following proposition, which gives some useful characterizations of the extensions of uniform maps.

Proposition 5.9. *Let, for some monotonicity type μ, f be an μ-monotone lattice map between $\mathbb{L}$ and $\mathbb{M}$. Then*
(1) *f° is the largest $(\rho, \gamma^\uparrow)$-continuous extension of f,*
(2) *$f^\bullet$ is the smallest $(\rho, \gamma^\downarrow)$-continuous extension of f,*
(3) *f is smooth iff f° is $(\rho, \gamma^\downarrow)$-continuous iff $f^\bullet$ is $(\rho, \gamma^\uparrow)$-continuous.*

Proof. We will only provide the proof for the case that f is order preserving, the more general case can easily be derived from this by the principle of order duality.

For the first part of the proposition, we already know by earlier results that f° is a $(\rho, \gamma^\uparrow)$-continuous lattice map that agrees with f on L. In order to prove that f is the largest such map, we need the following property of the Scott topology:

$$\overline{L} \setminus \downarrow u \text{ is Scott open for any } u \text{ in } \overline{\mathbb{L}}. \tag{5}$$

For a proof of this, let D be an up-directed set in $\overline{\mathbb{L}}$ such that $\bigvee D \in \overline{L} \setminus \downarrow u$. Suppose for contradiction that $D \cap (\overline{L} \setminus \downarrow u) = \varnothing$, that is, $D \subseteq \downarrow u$. But then u is an upper bound of D, so that $\bigvee D \leq u$, which gives the desired contradiction $\bigvee D \in \downarrow u$. This proves (5).

Now let $g : \overline{\mathbb{L}} \to \overline{\mathbb{M}}$ be some $(\rho, \gamma^\uparrow)$-continuous extension of f, but suppose for contradiction that $g(x) \not\leq f^\circ(x)$ for some $x \in \overline{L}$. By (5), the set $\overline{M} \setminus \downarrow f^\circ(x)$ is a $\gamma^\uparrow$-open neighborhood of $g(x)$. Hence by $(\rho, \gamma^\uparrow)$-continuity there is some ρ-basic open $[a, b]$ around x with $g[a, b] \subseteq \overline{M} \setminus \downarrow f^\circ(x)$; in particular, we have that $g(a) \not\leq f^\circ(x)$. But $g(a) = f(a)$, and as f is order preserving, so is f°. This gives $g(a) = f(a) \leq f^\circ(x)$, which provides the desired contradiction.

The second part of the Proposition follows by order duality, and the last part is immediate from the earlier parts and the fact that $f^\circ \leq f^\bullet$. □

For arbitrary, non-uniform lattice maps, the statements of the previous Proposition need not hold, as the following example witnesses.

Example 5.10. Here we show the existence of a (non-uniform) lattice map $f : \mathbb{D} \to \mathbb{E}$ and a $(\rho, \gamma^\uparrow)$-continuous extension $g : \overline{\mathbb{D}} \to \overline{\mathbb{E}}$ such that $g \not\leq f^\circ$.

Take the lattice $\mathbb{D}$ of Example 3.11, and let $\mathbb{E}$ be the lattice depicted in Figure 4. $\mathbb{E}$ is complete and hence coincides with its MacNeille completion: $\mathbb{E} = \overline{\mathbb{E}}$. Consider the map $f : D \to E$ given by

$$f(x) := \begin{cases} a_n & \text{if } x = n \in \omega, \\ b_n & \text{if } x = \breve{n} \in \breve{\omega}. \end{cases}$$

Clearly f is not uniform, so for the definition of $f^\circ(\infty)$, we have to consider intervals surrounding ∞. Given such an interval, say $[n, \breve{m}]$ with $n, m \in \omega$, both n and $\breve{m}$ belong to the set $D_{[n,\breve{m}]} = D \cap [n, \breve{m}]$. From this it follows that $f[D_{[n,\breve{m}]}]$ contains both a_n and b_m, whence $\bigwedge f[D_{[n,\breve{m}]}] = \bot$. Thus we find that $f^\circ(\infty) = \bigvee\{\bigwedge f[D_{[n,\breve{m}]}] \mid n, m \in \omega\} = \bot$.

Now consider the map g, defined as the extension of f mapping ∞ to the element t_∞. Clearly then we have that $g(\infty) \not\leq f^\circ(\infty)$, so that it is left to show that g is

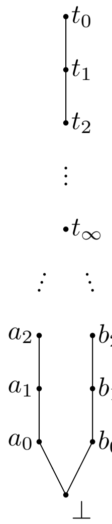

FIGURE 4. the lattice $\mathbb{E}$.

$(\rho, \gamma^{\uparrow})$-continuous. The key observation here is that any Scott open set of $\mathbb{E}$ which contains t_∞, must also contain elements a_k and b_m for some $k, m \in \omega$. From this it follows that every Scott open set of $\mathbb{E}$ must have one of the following forms: $\varnothing$, E, ${\uparrow}t_k$ with $k \in \omega$, or ${\uparrow}a_k \cup {\uparrow}b_m$ with $k, m \in \omega$. We have $g^{-1}(\varnothing) = \varnothing$, $g^{-1}(E) = D$, $g^{-1}({\uparrow}t_k) = \varnothing$, and $g^{-1}({\uparrow}a_k \cup {\uparrow}b_m) = [k, \breve{m}]$. From this the $(\rho, \gamma^{\uparrow})$-continuity of g is immediate.

Continuity properties. We will now see how some important properties of lattice maps are reflected as continuity properties. We start with the lattice operations themselves.

Proposition 5.11. *Let $\mathbb{L}$ be a lattice. Then*

(1) *the join operation $\overline{\vee}$ on $\overline{\mathbb{L}}$ is both $(\gamma^{\uparrow}, \gamma^{\uparrow})$- and $(\rho^{\downarrow}, \rho^{\downarrow})$-continuous,*
(2) *the meet operation $\overline{\wedge}$ on $\overline{\mathbb{L}}$ is both $(\gamma^{\downarrow}, \gamma^{\downarrow})$- and $(\rho^{\uparrow}, \rho^{\uparrow})$-continuous,*

Proof. Since $\overline{\vee}$ is completely additive, it certainly preserves directed joins, that is, it is Scott continuous. For $(\rho^{\downarrow}, \rho^{\downarrow})$-continuity, take some basic open set, say, ${\downarrow}b$, in the topology $\rho^{\downarrow}_M$. It is easy to check that

$$\overline{\vee}^{-1}[{\downarrow}b] = \{(x, y) \in \overline{L} \times \overline{L} \mid x \overline{\vee} y \leq b\} = \overline{\mathbb{L}}_{\downarrow b} \times \overline{\mathbb{L}}_{\downarrow b} = \overline{\mathbb{L}}^2_{\downarrow(b,b)}.$$

Clearly then, $\overline{\vee}^{-1}[{\downarrow}b]$ is a basic open set of the $\rho^{\downarrow}$ topology on $(\overline{\mathbb{L}})^2 = \overline{\mathbb{L}^2}$, and so it follows that $\overline{\vee}$ is $(\rho^{\downarrow}, \rho^{\downarrow})$-continuous. The results for the meet operation are then immediate by order duality. $\square$

Next, we turn to continuity properties of residuated maps. First we consider maps that are left- and right residuated, such as the conjunction in a Heyting algebra.

Proposition 5.12. *Let $f : \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{M}$ be both left- and right residuated. Then f° is $(\gamma^\uparrow, \gamma^\uparrow)$-continuous.*

Proof. If f is both left- and right residuated, then so is f°, cf. Proposition 3.17. Then it follows by standard results that f° is a complete operator, and from this it is immediate that f° preserves all directed joins, and hence, is Scott continuous. □

In case we are dealing with a (unary) residuated map, we can prove an even stronger, and very useful, result. Recall from Proposition 3.19 that both residuated maps and their residuals are smooth.

Proposition 5.13. *Let $f : \mathbb{L} \to \mathbb{M}$ be a residuated map between the lattices $\mathbb{L}$ and $\mathbb{M}$, with residual $g : \mathbb{M} \to \mathbb{L}$. Then*
(1) *$\overline{f}$ is both $(\gamma^\uparrow, \gamma^\uparrow)$ and $(\rho^\downarrow, \rho^\downarrow)$-continuous,*
(2) *$\overline{g}$ is both $(\gamma^\downarrow, \gamma^\downarrow)$ and $(\rho^\uparrow, \rho^\uparrow)$-continuous.*

Proof. Assume that the lattice maps $f : \mathbb{L} \to \mathbb{M}$ and $g : \mathbb{M} \to \mathbb{L}$ form a residual pair, let x be an arbitrary element of $\overline{L}$, and take some $\rho^\downarrow_M$-basic neighborhood of $\overline{f}(x)$, say, $\downarrow b$ with $b \in M$. Recall from Proposition 3.19 that $\overline{f}$ and $\overline{g}$ form a residual pair. Hence we find

$$\begin{aligned}(\overline{f})^{-1}[\downarrow b] &= \{x \in \overline{L} \mid \overline{f}(x) \leq b\} \\ &= \{x \in \overline{L} \mid x \leq \overline{g}(b) = g(b)\} \\ &= \downarrow g(b).\end{aligned}$$

In other words, we see that the inverse image of $\downarrow b$ under $\overline{f}$ is the ρ_L-open set $\downarrow g(b)$. This proves the $(\rho^\downarrow, \rho^\downarrow)$-continuity of $\overline{f}$. The $(\gamma^\uparrow, \gamma^\uparrow)$-continuity of $\overline{f}$ is a direct consequence of its complete additivity, and the results for g follow by order duality. □

As a corollary, we see that if f is *both* residuated *and* a residual, then $\overline{f}$ is a complete homomorphism, and (τ, τ)-continuous for every $\tau \in \{\rho^\uparrow, \rho^\downarrow, \rho, \gamma^\uparrow, \gamma^\downarrow, \gamma\}$. As a particular instance of this, we mention the following fact (see Example 3.21 for the residuation properties of projection maps).

Proposition 5.14. *Let $\mathbb{L}_1, \ldots, \mathbb{L}_n$ be lattices. Then each projection operation $\overline{\pi_i} : \overline{\mathbb{L}_1} \times \cdots \times \overline{\mathbb{L}_n} \to \overline{\mathbb{M}}$ is (τ, τ)-continuous for all $\tau \in \{\gamma^\uparrow, \gamma^\downarrow, \gamma, \rho^\uparrow, \rho^\downarrow, \rho\}$.*

Products and composition. In the next section, where we discuss *term functions*, we will need some results concerning the continuity properties of *product maps* and *compositions* of maps. We start with products.

Definition 5.15. Let $\{f_i : A \to B_i\}$ be a collection of maps, then the *product map* $\langle f_i \rangle_{i \in I} : A \to \prod_{i \in I} B_i$ is given by

$$\langle f_i \rangle_{i \in I}(a)(i) := f_i(a).$$

In case I is the set $\{1, \ldots, n\}$, this product map is denoted as $\langle f_1, \ldots, f_n \rangle$. ◁

Proposition 5.16. *Let $\{f_i : \mathbb{L} \to \mathbb{M}_i \mid i \in I\}$ be a family of lattice maps. Then*

$$(\langle f_i \rangle_{i \in I})^\circ = \langle f_i^\circ \rangle_{i \in I}. \tag{6}$$

Proof. Straightforward by the definitions and the observation that we may take $\prod \overline{\mathbb{M}_i}$ as the MacNeille completion of the product $\prod \mathbb{M}_i$. $\square$

Proposition 5.17. *Let $\mathbb{L}$ and, for each i with $1 \leq i \leq n$, $\mathbb{M}_i$ lattices, and let, for each i, $f_i : \overline{\mathbb{L}} \to \overline{\mathbb{M}}_i$ be some lattice map. Then for each topology τ from the set $\{\gamma^\uparrow, \gamma^\downarrow, \rho^\uparrow, \rho^\downarrow, \rho\}$, the following holds:*

$$\langle f_1, \ldots, f_n \rangle \text{ is } (\tau, \tau)\text{-continuous iff each } f_i \text{ is } (\tau, \tau)\text{-continuous.} \tag{7}$$

Proof. We only prove the proposition for the cases that $\tau = \gamma^\uparrow$ and $\tau = \rho^\uparrow$, since the remaining cases are immediate consequences of this. Also, we only consider the direction from right to left in (7), since the other direction can easily be derived from properties of the projection operators given in Proposition 5.14. Let f denote the map $\langle f_1, \ldots, f_n \rangle$, and let $\mathbb{M}$ be the lattice $\prod \mathbb{M}_i$, then $\overline{\mathbb{M}}$ represents the product $\prod \overline{\mathbb{M}}_i$.

First suppose that each $f_i : \overline{\mathbb{L}} \to \overline{\mathbb{M}}_i$ is $(\rho^\uparrow, \rho^\uparrow)$-continuous. Since the set $\{\uparrow a \subseteq \overline{M} \mid a \in M\}$ forms a *basis* for $\rho^\uparrow_M$, in order to prove that $f : \overline{\mathbb{L}} \to \overline{\mathbb{M}}$ is also $(\rho^\uparrow, \rho^\uparrow)$-continuous, it suffices to show that $f^{-1}[\uparrow a] \in \rho^\uparrow_L$ for each $a = (a_1, \ldots, a_n)$ in $\mathbb{M}$. But it follows immediately from the definitions that $f^{-1}[\uparrow a] = \bigcap_{1 \leq i \leq n} f_i^{-1}[\uparrow a_i]$. From this the result is immediate, since each set $f_i^{-1}[\uparrow a_i]$ is $\rho^\uparrow_L$-open by the assumed $(\rho^\uparrow, \rho^\uparrow)$-continuity of f_i.

The result for $\tau = \gamma^\uparrow$ easily follows from the characterization of Scott continuity as the preservation of upward directed joins. That is, suppose that each $f_i : \overline{\mathbb{L}} \to \overline{\mathbb{M}}_i$ preserves upward directed joins, and let $D \subseteq \overline{L}$ be an upward directed set. Since by the assumption on the f_i we have that

$$f(\bigvee D) = (f_1(\bigvee D), \ldots, f_n(\bigvee D)) = (\bigvee f_1[D], \ldots, \bigvee f_n[D]),$$

in order to show that f preserves upward directed joins, it suffices to show that

$$(\bigvee f_1[D], \ldots, \bigvee f_n[D]) = \bigvee f[D].$$

This proof is completely straightforward, and hence we leave it for the reader. $\square$

Remark 5.18. We do not know whether Proposition 5.17 applies to (γ, γ)-continuity as well. In case the maps involved are μ-monotone, a fairly straightforward adaptation of the proof for $(\gamma^\uparrow, \gamma^\uparrow)$-continuity can be used to prove that indeed, (7) also holds in case $\tau = \gamma$.

We finish this section with an investigation of the interaction between the operation of composing two lattice maps, and that of taking their MacNeille extensions. That is, we will take a look at the relation between the maps $(gf)^\circ$ and $g^\circ f^\circ$ for maps $f : \mathbb{K} \to \mathbb{L}$, and $g : \mathbb{L} \to \mathbb{M}$ (and likewise for the upper extensions). We are obviously eager to find cases in which we have $(gf)^\circ = g^\circ f^\circ$, but also conditions under which one of the inequalities ($\leq$ or $\geq$) apply will turn out to be of interest. As we will see shortly, many of these conditions can naturally be described in *topological* terms.

Our main tool will be the following result.

Proposition 5.19. *Let $f : \mathbb{K} \to \mathbb{L}$ and $g : \mathbb{L} \to \mathbb{M}$ be isotone lattice maps. Then*

(1) $(gf)^\circ \leq g^\circ f^\circ \leq \left\{ \begin{array}{c} g^\circ f^\bullet \\ g^\bullet f^\circ \end{array} \right\} \leq g^\bullet f^\bullet \leq (gf)^\bullet$,
(2) $(gf)^\circ \geq g^\circ f^\circ$ *whenever* $g^\circ f^\circ$ *is* $(\rho, \gamma^\uparrow)$*-continuous,*
(3) $(gf)^\bullet \leq g^\bullet f^\bullet$ *whenever* $g^\bullet f^\bullet$ *is* $(\rho, \gamma^\downarrow)$*-continuous.*

Proof. We start with the first inequality of the first part of the proposition. By definition it holds that

$$(gf)^\circ(x) = \bigvee\{gf(a) \mid x \geq a \in K\} \leq \bigvee\{g(b) \mid f^\circ(x) \geq b \in L\} = g^\circ(f^\circ(x)).$$

Here the first and last step are by definition of $(gf)^\circ$ and of g°, respectively. The crucial second step follows from the fact that monotonicity of f implies that $f(a) \leq f^\circ(x)$ whenever $a \leq x$, so that every joinand of $\bigvee\{gf(a) \mid x \geq a \in K\}$ also occurs in $\bigvee\{g(b) \mid f^\circ(x) \geq b \in L\}$. The last inequality of part 1 immediately follows from this by the principle of order duality. The other inequalities are consequences of the fact that MacNeille extensions of isotone maps are isotone, and the inequalities $f^\circ \leq f^\bullet$ and $g^\circ \leq g^\bullet$.

The second and third part of the proposition are straightforward corollaries of Proposition 5.9. □

6. Preservation of equations

6.1. Introduction. In this section we address the question which (equational) properties of lattice expansions are preserved under taking MacNeille completions. Clearly, we have to start with making this question more precise, since it will be clear from the previous section that there are *various* ways to complete a lattice expansion. Given an $\mathcal{E}$-expanded lattice $\mathbb{A}$, for each function symbol $\nabla \in \mathcal{E}$, one may take the lower or the upper extension of $\nabla^{\mathbb{A}}$; to distinguish the various possible extensions of $\mathbb{A}$, we introduce the notion of an *extension type*.

Definition 6.1. Given an expansion type $\mathcal{E}$, an *extension type* ψ is a map $\psi : \mathcal{E} \to \{\circ, \bullet\}$. The ψ*-MacNeille extension* of an $\mathcal{E}$-expanded lattice $\mathbb{A}$ is the $\mathcal{E}$-expanded lattice $\mathbb{A}^\psi$ of which the lattice reduct $(\mathbb{A}^\psi)_\flat$ is the MacNeille completion of $\mathbb{A}_\flat$, while for $\nabla \in \mathcal{E}$ we define

$$\nabla^{\mathbb{A}^\psi} := (\nabla^{\mathbb{A}})^{\psi(\nabla)}.$$

That is, ψ determines whether we take the lower or the upper extension of $\nabla^{\mathbb{A}}$. In case we have $\psi(\nabla) = \circ$ for all $\nabla \in \mathcal{E}$ we write $\mathbb{A}^\circ$ rather than $\mathbb{A}^\psi$; this structure is called the *lower MacNeille extension* of $\mathbb{A}$. A similar definition applies to the *upper MacNeille extension* $\mathbb{A}^\bullet$.

An $\mathcal{E}$-expanded lattice $\mathbb{A}$ is called *smooth* if it interprets each of the function symbols in $\mathcal{E}$ as a smooth operation. Such an algebra has a unique MacNeille extension, which will be denoted as $\overline{\mathbb{A}}$. ◁

Definition 6.2. Let K be some class of lattice expansions, and let P be some property of lattice expansions. Then we say that P is *MacNeille* ψ*-canonical on* K if it is preserved under taking MacNeille ψ-completions; that is, if $\mathbb{A}^\psi$ has property P for every algebra $\mathbb{A}$ in K that has property P. We restrict our attention to properties that are definable by equations. In accordance with the just defined notion, we say that an equation $s \approx t$ is *MacNeille ψ-canonical on* K, if $\mathbb{A}^\psi \models s \approx t$ for all algebras $\mathbb{A}$ in K such that $\mathbb{A} \models s \approx t$.

The notions of *lower* and *upper MacNeille canonicity* are defined analogously. In case K consists of smooth algebras, all versions of MacNeille canonicity coincide, and we will speak of *MacNeille canonicity* per se. $\lhd$

Two explanatory, or motivating, remarks may be in order. First, we made the definition of MacNeille canonicity relative to a base class K of lattice expansions. This is not without reason: as we will see, the better the class K fares under taking a certain type of MacNeille completions, the more equations are MacNeille canonical for that type. Now in general, it will be *undecidable* whether a given equation is MacNeille canonical (for any fixed extension type) on some given class K. (For varieties of modal algebras, this can be proved using standard methods from modal logic, see Kracht & Wolter [33, page 137]). This already indicates that the general picture may be rather complex, and that we cannot hope to give a comprehensive survey here. In many cases however, fairly sharp sufficient conditions can be formulated.

Second, while in this paper we will generally restrict our attention to lower and upper extension type (that is, the ones in which we uniformly take the lower respectively the upper extension of each additional operation), there are situations in which it is more natural to consider mixed extension types. For instance, the only extension type ψ for which the variety of lattices with residuated binary operations is MacNeille ψ-canonical, is the one in which we take the lower extension of the residuated operation, and the upper extension of the residuated one.

It seems sensible to restrict our attention to a context of lattice expansions that are, themselves, already closed under taking MacNeille completions. In section 3 we identified two reasons why a class of lattice expansions is closed under taking MacNeille completions: it may be either because the additional operations manifest some kind of residuation properties, or because the lattice reducts satisfy some kind of distributive laws. Taking the class of Boolean algebras as an example of the latter case, we arrive at the following choice of classes of order preserving lattice expansions to be studied in this section:

- arbitrary lattices with arbitrary order preserving operations;
- expansions of Boolean algebras;
- (arbitrary) lattices expanded with operations that are all (left/right) residuated.

After that, we also briefly look at lattices that are expanded with operations that are not necessarily order preserving, but still uniform.

Before we turn to these individual classes, let us briefly describe our method for proving canonicity in general. This method ultimately goes back to the seminal paper by Jónsson & Tarski on canonical extensions of Boolean algebras with operators; much of our notation and terminology is taken from Jónsson [30]. Consider an $\mathcal{E}^+$-term $t(x_1, \ldots, x_n)$. Recall that on each $\mathcal{E}^+$-algebra $\mathbb{A}$, t gives rise to a *term function* $t^{\mathbb{A}} : A^n \to A$, and that the validity of an equation $s \approx t$ can be formulated using these terms functions:

$$\mathbb{A} \models s \approx t \text{ iff } s^{\mathbb{A}} = t^{\mathbb{A}}. \tag{8}$$

Hence, for the ψ-MacNeille extension $\mathbb{A}^{\psi}$, we find that

$$\mathbb{A}^{\psi} \models s \approx t \text{ iff } s^{\mathbb{A}^{\psi}} = t^{\mathbb{A}^{\psi}}. \tag{9}$$

However, it trivially follows from (8) that, for instance,

$$\mathbb{A} \models s \approx t \text{ iff } (s^{\mathbb{A}})^{\circ} = (t^{\mathbb{A}})^{\circ}. \tag{10}$$

Hence, in order to prove the ψ-MacNeille canonicity of the equation $s \approx t$, it would suffice to prove that $s^{\mathbb{A}^{\psi}} = (s^{\mathbb{A}})^{\circ}$ and $t^{\mathbb{A}^{\psi}} = (t^{\mathbb{A}})^{\circ}$. This motivates a careful analysis of the relation between functions of the form $t^{\mathbb{A}^{\psi}}$ (the term function of t in $\mathbb{A}^{\psi}$, and those of the form $(t^{\mathbb{A}})^{\circ}$ and $(t^{\mathbb{A}})^{\bullet}$ (the lower and upper extension of the term function $t^{\mathbb{A}}$). Such an analysis will be carried out in the sequel; here we restrict our attention to the extensions $\mathbb{A}^{\circ}$ and $\mathbb{A}^{\bullet}$.

Definition 6.3. Let $\mathcal{E}$ be some lattice expansion type, and K some class of $\mathcal{E}$-expanded lattices. We call an $\mathcal{E}^{+}$-term t *lower expanding* on K if $(t^{\mathbb{A}})^{\circ} \leq t^{\mathbb{A}^{\circ}}$ for all $\mathbb{A}$ in K, *lower contracting* if $(t^{\mathbb{A}})^{\circ} \geq t^{\mathbb{A}^{\circ}}$ for all $\mathbb{A}$ in K, and *lower stable* if $(t^{\mathbb{A}})^{\circ} = t^{\mathbb{A}^{\circ}}$ for all $\mathbb{A}$ in K. Similar definitions apply to the notions of upper expanding, upper contracting, and upper stable. $\lhd$

It is important to realize that, even if we are dealing with a smooth algebra $\mathbb{A}$, the upper and the lower version of the just defined notions need not coincide. Indeed, we will encounter examples of terms t and smooth algebras $\mathbb{A}$ such that $(t^{\mathbb{A}})^{\circ} = t^{\overline{\mathbb{A}}} \neq (t^{\mathbb{A}})^{\bullet}$. That is, smoothness of the primitive operations does not guarantee smoothness of the term functions.

Our last general definition concerns syntax. It will come in very handy in the remainder of this section, when we discuss properties of various kinds of terms and equations.

Definition 6.4. Let $\mathcal{E}$ be some lattice expansion type. Given subsets $\mathcal{F}, \mathcal{F}'$ of $\mathcal{E}^{+}$, we define an *$\mathcal{F}$-term* to be any $\mathcal{E}^{+}$-term that only uses symbols from $\mathcal{F}$. An $\mathcal{F}/\mathcal{F}'$-term is a term of the form $s(u_1, \ldots, u_n)$ such that s is an $\mathcal{F}$-term and each u_i, an $\mathcal{F}'$-term. A *lattice term* is simply a $\{\vee, \wedge, \bot, \top\}$-term, and a term is *basic* if it has at most one occurrence of a function symbol. $\lhd$

Observe that both $\mathcal{F}$ and $\mathcal{F}'$-terms are special instances of $\mathcal{F}/\mathcal{F}'$-terms, and that variables are $\mathcal{F}$-terms for any set $\mathcal{F}$.

6.2. Order preserving lattice expansions. We start our discussion of MacNeille canonical equations with the class of order preserving lattice expansions.

Definition 6.5. For a lattice expansion type $\mathcal{E}$ we let $\mathsf{MLE}_{\mathcal{E}}$ denote the class of $\mathcal{E}$-expanded lattices in which every function symbol ∇ in $\mathcal{E}$ is interpreted as an order preserving operation. $\lhd$

About this rather general class there is not much to be said. The following proposition states some useful results concerning terms.

Proposition 6.6. *Let $\mathcal{E}$ be some lattice expansion type, and let t be some $\mathcal{E}^+$-term. Then*

1. *$t^{\mathbb{A}}$ is order preserving for any lattice expansion $\mathbb{A}$ in $\mathsf{MLE}_{\mathcal{E}}$;*
2. *t is lower expanding and upper contracting on $\mathsf{MLE}_{\mathcal{E}}$;*
3. *for any $\mathbb{A}$ in $\mathsf{MLE}_{\mathcal{E}}$: if $t^{\mathbb{A}^\circ}$ is $(\rho, \gamma^\uparrow)$-continuous, then t is lower stable on $\mathbb{A}$, and if $t^{\mathbb{A}^\bullet}$ is $(\rho, \gamma^\downarrow)$-continuous, then t is upper stable on $\mathbb{A}$.*

Proof. The first part of the proposition follows by a straightforward term induction. Concerning the second part, we only show that $\mathcal{E}^+$-terms are lower expanding; the statement on upper contraction follows by order duality. The proof that $\mathcal{E}^+$-terms are lower expanding, proceeds via a term induction, of which we only treat the induction step. That is, let t be the term $\nabla(s_1, \ldots, s_n)$, and let $\mathbb{A}$ be an $\mathcal{E}$-expanded lattice in which all the primitive function symbols are interpreted as order preserving operations. Then we have

$$\begin{array}{rcl}
(t^{\mathbb{A}})^\circ & = & (\nabla^{\mathbb{A}} \circ \langle s_1^{\mathbb{A}}, \ldots, s_n^{\mathbb{A}} \rangle)^\circ \\
& \leq & (\nabla^{\mathbb{A}})^\circ \circ \langle s_1^{\mathbb{A}}, \ldots, s_n^{\mathbb{A}} \rangle^\circ \\
& = & \nabla^{\mathbb{A}^\circ} \circ \langle (s_1^{\mathbb{A}})^\circ, \ldots, (s_1^{\mathbb{A}})^\circ \rangle \\
& \leq & \nabla^{\mathbb{A}^\circ} \circ \langle s_1^{\mathbb{A}^\circ}, \ldots, s_n^{\mathbb{A}^\circ} \rangle \\
& = & t^{\mathbb{A}^\circ}.
\end{array}$$

Here the first and last steps are by definition, the second step is by Proposition 5.19 (using the fact that both $(\nabla^{\mathbb{A}})^\circ : A^n \to A$ and $\langle s_1^{\mathbb{A}}, \ldots, s_n^{\mathbb{A}} \rangle^\circ : A^n \to A^n$ are order preserving maps), the third step is by Proposition 5.16, and the fourth step is by the inductive hypothesis and the monotonicity of the map $\nabla^{\mathbb{A}^\circ}$.

For the final statement of the proposition, consider some algebra $\mathbb{A}$ in $\mathsf{MLE}_{\mathcal{E}}$. It follows from the first part of the proposition that $t^{\mathbb{A}}$ is order preserving, so that by Proposition 5.9, $(t^{\mathbb{A}})^\circ$ is the *largest* $(\rho, \gamma^\uparrow)$-continuous extension of $t^{\mathbb{A}}$. Now it is easy to see that $t^{\mathbb{A}^\circ}$ is an extension of $t^{\mathbb{A}}$ as well, so if $t^{\mathbb{A}^\circ}$ is $(\rho, \gamma^\uparrow)$-continuous, then we have $t^{\mathbb{A}^\circ} \leq (t^{\mathbb{A}})^\circ$. This proves that t is lower contracting on $\mathbb{A}$, so with the second part of the proposition we see that t is in fact lower stable. □

When it comes to MacNeille canonicity of equations, we can prove the following result.

Theorem 6.7. *Let $\mathcal{E}$ be some lattice expansion type, let ∇ be some operation symbol in $\mathcal{E}$, and let t be some $\mathcal{E}^+$-term. Then*

1. *the inequality $\nabla(x_1, \ldots, x_n) \preceq t$ is lower MacNeille canonical on $\mathsf{MLE}_{\mathcal{E}}$;*
2. *the inequality $t \preceq \nabla(x_1, \ldots, x_n)$ is upper MacNeille canonical on $\mathsf{MLE}_{\mathcal{E}}$.*

Proof. We only prove the first part of the proposition, the second part is then immediate by order duality. Let $\mathbb{A}$ be an order preserving $\mathcal{E}$-expanded lattice such that $\mathbb{A} \models \nabla(x_1, \ldots, x_n) \preceq t$, that is, $\nabla^{\mathbb{A}} \leq t^{\mathbb{A}}$. Then $\nabla^{\mathbb{A}^\circ} = (\nabla^{\mathbb{A}})^\circ \leq (t^{\mathbb{A}})^\circ \leq t^{\mathbb{A}^\circ}$. Here the first identity is by definition, the first inequality is a basic property of the lower MacNeille extension, and the last inequality is by Proposition 6.6. □

Example 6.8. A *closure operation* on a lattice is an order preserving map satisfying the inequalities $x \preceq \nabla x$ and $\nabla\nabla x \preceq \nabla x$. It is then an immediate consequence of Theorem 6.7 that the upper extension of a closure operation is again a closure operation.

As far as we know, not much more can be proved in the full generality of isotone lattice expansions.

Example 6.9. For instance, consider the inequality $\nabla(x \vee y) \preceq \nabla x \vee \nabla y$, which looks like a rather innocent extension of the examples covered by Theorem 6.7. Note that in the case that ∇ is order preserving, this inequality states that ∇ is *additive*. From the Examples 3.8 and 3.10 we may then derive that the given inequality is neither lower nor upper MacNeille canonical.

6.3. Expansions of Boolean algebras. In section 3 we saw that in case the base lattice satisfies some infinite distributive law, more properties of the additional operations may be preserved under taking one of the MacNeille completions (but not necessarily under the other). We now investigate this phenomenon in some more detail, and from an equational perspective. For convenience we restrict our attention to expansions of Boolean algebras (although similar results can be obtained for expanded Heyting algebras).

Definition 6.10. For a lattice expansion type $\mathcal{E}$, let $\mathsf{MBAE}_{\mathcal{E}}$ denote the class of $\mathcal{E}$-expanded Boolean algebras in which every function symbol ∇ in $\mathcal{E}$ is interpreted as an order preserving operation. $\lhd$

The key technical result is the following. Recall that a term is basic if it has at most one occurrence of a function symbol.

Proposition 6.11. *Let $\mathcal{E}$ be some lattice expansion type. Then*
1. *conjunctions of basic terms are lower stable on* $\mathsf{MBAE}_{\mathcal{E}}$*,*
2. *disjunctions of basic terms are upper stable on* $\mathsf{MBAE}_{\mathcal{E}}$*.*

Proof. By Proposition 6.6, in order to prove the first part of the proposition, it suffices to show that for an arbitrary conjunction of basic terms t, the map $t^{\mathbb{A}^\circ}$ is $(\rho^\uparrow, \gamma^\uparrow)$-continuous, for an arbitrary $\mathcal{E}$-expanded Boolean algebra $\mathbb{A}$. Consider such a term t; for simplicity we may assume without loss of generality that t is of the form $\nabla_1(x_1, \ldots, x_n) \wedge \nabla_2(x_1, \ldots, x_n)$. Then it follows that $t^{\mathbb{A}^\circ}$ is given as the map $\wedge^{\mathbb{A}^\circ} \circ \langle \nabla_1^{\mathbb{A}^\circ}, \nabla_2^{\mathbb{A}^\circ} \rangle$. Now $\wedge^{\mathbb{A}^\circ}$ is $(\gamma^\uparrow, \gamma^\uparrow)$-continuous by Proposition 5.12, while $\langle \nabla_1^{\mathbb{A}^\circ}, \nabla_2^{\mathbb{A}^\circ} \rangle = \langle (\nabla_1^{\mathbb{A}})^\circ, (\nabla_2^{\mathbb{A}})^\circ \rangle = \langle \nabla_1^{\mathbb{A}}, \nabla_2^{\mathbb{A}} \rangle^\circ$ by definition and Proposition 5.16, and so $\langle \nabla_1^{\mathbb{A}^\circ}, \nabla_2^{\mathbb{A}^\circ} \rangle$ is $(\rho, \gamma^\uparrow)$-continuous by Proposition 5.6. From this it follows that $t^{\mathbb{A}^\circ}$ is the composition of a $(\gamma^\uparrow, \gamma^\uparrow)-$ and a $(\rho, \gamma^\uparrow)$-continuous map, and hence, $(\rho, \gamma^\uparrow)$-continuous, as required.

The second part of the proposition is immediate by order duality. $\square$

As an immediate consequence of this proposition we can prove the following extension of Theorem 6.7.

Theorem 6.12. *Let $\mathcal{E}$ be some lattice expansion type, and let t be some $\mathcal{E}^+$-term. Then on $\mathsf{MBAE}_{\mathcal{E}}$:*

(1) *inequalities of the form $s \preceq t$, with s a conjunction of basic terms, are lower MacNeille canonical;*
(2) *inequalities of the form $t \preceq s$, with s a disjunction of basic terms, are upper MacNeille canonical.*

Proof. Let $\mathcal{E}$, s and t be as in part 1 of the Theorem, and let $\mathbb{A}$ be some $\mathcal{E}$-expanded Boolean algebra such that $\mathbb{A} \models s \preceq t$. It follows from Proposition 6.11 that s is lower stable on $\mathbb{A}$ and from Proposition 6.6 that t is upper contracting. From this we see that $s^{\mathbb{A}^\circ} \leq (s^{\mathbb{A}})^\circ \leq (t^{\mathbb{A}})^\circ \leq t^{\mathbb{A}^\circ}$, so that $\mathbb{A}^\circ \models s \preceq t$, as required. □

Example 6.13. It easily follows from the above theorem that the inequality $\nabla(x \vee y) \preceq \nabla x \vee \nabla y$ is preserved under taking upper MacNeille completions of Boolean algebras. This explains the upper MacNeille canonicity of the property of additivity, and reveals why the variety of modal algebras is upper MacNeille canonical.

Another example of an upper McN-canonical equation is $\nabla\nabla x \preceq x \vee \nabla x$.

Unfortunately however, the fact that we are dealing with Boolean algebras does not allow for much improvement on Theorem 6.12, unless we consider *residuated* Boolean algebra expansions, see [24] (or the next subsection). More precisely, we are not aware of any example of a MacNeille canonical variety of Boolean algebra expansions, that is not covered by Theorem 6.12 or Theorem 6.33.

Also, counterexamples abound if we weaken the assumptions of Theorem 6.12. For instance, we already saw in Example 3.8 that additivity is not preserved under taking lower MacNeille completions, even if the base lattice is a Boolean algebra. And in the case that we focus on upper completions, and assume that all of the additional operations are unary operators, there is not much good news to report either. Not even the *strictly positive equations* (that is, equations not involving the negation symbol at all) are upper MacNeille canonical, as the following example, which was obtained in cooperation with Guram and Nick Bezhanishvili and Johh Harding, witnesses.

Example 6.14. Let Lin be the inequality

$$\Diamond x \wedge \Diamond y \preceq \Diamond(x \wedge y) \vee \Diamond(\Diamond x \wedge y) \vee \Diamond(x \wedge \Diamond y).$$

Now let $\mathbb{B}$ be the Boolean algebra consisting of the finite and cofinite sets of natural numbers, and consider the following interpretation of $\Diamond$ on $\mathbb{B} \times \mathbf{2}$:

$$\Diamond(a,b) := \begin{cases} (a,b) & \text{if } a \text{ is finite} \\ (a,1) & \text{if } a \text{ is cofinite} \end{cases}$$

It is then a tedious but straightforward exercise to show that this gives a normal and additive operation on $\mathbb{B} \times \mathbf{2}$, and that Lin is valid on the resulting modal algebra $\mathbb{A} = \langle \mathbb{B} \times \mathbf{2}, \Diamond \rangle$.

Clearly, $\mathbb{P}(\omega)$ — the power set algebra of the set ω of natural numbers is the MacNeille completion of $\mathbb{B}$, and we may take $\mathbb{P}(\omega) \times \mathbf{2}$ to be the MacNeille completion of the algebra $\mathbb{B} \times \mathbf{2}$. It is easy to see that the upper extension of $\Diamond$ is the map $\Diamond^\bullet$ on $\mathbb{P}(\omega) \times \mathbf{2}$ given by

$$\Diamond^\bullet(a,b) := \begin{cases} (a,b) & \text{if } a \text{ is finite} \\ (a,1) & \text{if } a \text{ is infinite} \end{cases}$$

Now let E be the set of even numbers, and O the set of odds. A straightforward calculation will reveal that the left and right hand side of Lin, when applied to the elements $x = (E,0)$ and $y = (O,0)$, will evaluate to the elements $(\varnothing,1)$ and $(\varnothing,0)$, respectively. This shows that Lin is not valid on the algebra $\overline{\mathbb{A}}$. Thus the validity of Lin is not preserved under taking upper MacNeille completions.

In passing, we note that it is an open problem whether lower Macneille canonicity implies upper MacNeille canonicity for modal algebras. That is, if V is a variety of modal algebras that is closed under taking lower MacNeille completions, is V then also closed under taking upper MacNeille completions?

6.4. Residuated lattice expansions. As was to be expected given the results in section 3, the best results on MacNeille canonicity of equations can be obtained for lattice expansions in which all primitive operations are either residuated or residuals. In the case of Boolean algebras with operators, this was known already from the work of Givant & Venema, who prove in [24] that every *Sahlqvist equation* is MacNeille canonical on the class of BAOs such that every non-Boolean primitive operation is residuated. We will see now (that is, here and in the next subsection), that this result does not depend on the Boolean nature of the base lattice.

Definition 6.15. Let $\mathcal{R}$ be an arbitrary but fixed lattice expansion type that can be divided into two sets $\{\Diamond_i \mid i \in I\}$ and $\{\Box_j \mid j \in J\}$ of unary operation symbols to be called *diamonds* and *boxes*, respectively.

A *residuated lattice expansion (of type $\mathcal{R}$)* is any $\mathcal{R}$-expanded lattice $\mathbb{A}$ such that every diamond is interpreted as a residuated operation, and every box, as a residual. The class of these algebras is denoted as $\mathsf{RLE}_{\mathcal{R}}$. ◁

In order to make our results as general as possible, when defining $\mathsf{RLE}_{\mathcal{R}}$ we imposed no requirements on the nature of the residuals of the interpreted diamonds, or on that of the operations residuated by the boxes. In particular, we do not demand that the residuals of the diamonds are given by the boxes, or term definable in some other way. Note however, that residuated lattice expansions are *smooth*. Hence, there is no distinction between the upper and lower versions of MacNeille canonicity. Recall that in this case we write $\overline{\mathbb{A}}$ for the unique MacNeille completion of a residuated lattice expansion $\mathbb{A}$.

Proposition 6.16. *Let t be an $\mathcal{R}$-term, and let $\mathbb{A}$ be some residuated lattice expansion for the similarity type $\mathcal{R}$. Then*

1. *if t is a $\{\Diamond,\vee\}$-term, then $t^{\overline{\mathbb{A}}}$ is both $(\gamma^\uparrow,\gamma^\uparrow)$- and $(\rho^\downarrow,\rho^\downarrow)$-continuous.*
2. *if t is a $\{\Box,\wedge\}$-term, then $t^{\overline{\mathbb{A}}}$ is both $(\gamma^\downarrow,\gamma^\downarrow)$- and $(\rho^\uparrow,\rho^\uparrow)$-continuous.*
3. *if t is a $\{\Diamond,\vee\}/\{\Box,\wedge\}$-term, then $t^{\overline{\mathbb{A}}}$ is $(\rho,\gamma^\uparrow)$-continuous.*

Proof. The first part of the Proposition can be proved via a straightforward induction on the complexity of $\{\Diamond, \vee\}$-terms. For the base step of this induction, where the term t is a variable, we use the fact that the projection operations extend to operations that are both $(\gamma^{\uparrow}, \gamma^{\uparrow})$ and $(\rho^{\downarrow}, \rho^{\downarrow})$-continuous (see Proposition 5.14). For the inductive step, first let t be of the form $t = u_1 \vee u_2$, and let τ denote either $\gamma^{\uparrow}$ or $\rho^{\downarrow}$. Then $t^{\overline{\mathbb{A}}} = (u_1 \vee u_2)^{\overline{\mathbb{A}}} = \vee^{\overline{\mathbb{A}}} \circ \langle u_1^{\overline{\mathbb{A}}}, u_2^{\overline{\mathbb{A}}} \rangle$. Now $\vee^{\overline{\mathbb{A}}}$ is (τ, τ)-continuous by Proposition 5.11, and $\langle u_1^{\overline{\mathbb{A}}}, u_2^{\overline{\mathbb{A}}} \rangle$ is (τ, τ)-continuous by the induction hypothesis and Proposition 5.17. But then $t^{\overline{\mathbb{A}}}$, being the composition of two (τ, τ)-continuous maps, is (τ, τ)-continuous itself. In the other inductive case, that is, with t of the form $\Diamond u$, we proceed in a similar way, now using Proposition 5.13 instead of Proposition 5.11.

The second part of the proposition follows by order duality. For the third part, suppose that t is of the form $s(u_1, \ldots, s_n)$, with s an $\{\Diamond, \vee\}$-term, and each u_i an $\{\Box, \wedge\}$-term. Then by the second part of the proposition, the map $\langle u_1^{\overline{\mathbb{A}}}, \ldots, u_n^{\overline{\mathbb{A}}} \rangle$ is $(\rho^{\uparrow}, \rho^{\uparrow})$-continuous, and thus, since $\gamma^{\uparrow} \subseteq \rho^{\uparrow}$ by Proposition 4.6, $(\rho^{\uparrow}, \gamma^{\uparrow})$-continuous. Hence, now using the first part of the proposition, we see that $t^{\overline{\mathbb{A}}} = s^{\overline{\mathbb{A}}} \circ \langle u_1^{\overline{\mathbb{A}}}, \ldots, u_n^{\overline{\mathbb{A}}} \rangle$ is the composition of a $(\gamma^{\uparrow}, \gamma^{\uparrow})$- and a $(\rho^{\uparrow}, \gamma^{\uparrow})$-continuous map, and thus itself $(\rho^{\uparrow}, \gamma^{\uparrow})$-continuous. Clearly, the key principle at work here is that of two *matching* continuities. $\Box$

As a corollary to Proposition 6.16 we obtain the following.

Proposition 6.17. *On the class $\mathsf{RLE}_{\mathcal{R}}$ of residuated lattice expansions:*

(1) *$\{\Diamond, \vee\}/\{\Box, \wedge\}$-terms are lower stable,*
(2) *$\{\Box, \wedge\}/\{\Diamond, \vee\}$-terms are upper stable,*
(3) *$\{\Diamond, \vee\}$ and $\{\Box, \wedge\}$-terms are both lower and upper stable.*

Proof. The first part of the proposition is an immediate corollary of the previous result, together with Proposition 6.6. The second part then follows through the principle of order duality, and the third part follows since the terms considered there all fall under the scope of both the first two parts. $\Box$

The two previous propositions yield the following harvest.

Theorem 6.18. *The following equations and inequalities are MacNeille canonical on the class $\mathsf{RLE}_{\mathcal{R}}$ of residuated lattice expansions:*

(1) *equations $s \approx t$ with s and t both $\{\Diamond, \vee\}/\{\Box, \wedge\}$-terms, or both $\{\Box, \wedge\}/\{\Diamond, \vee\}$-terms,*
(2) *inequalities $s \preceq t$ with s a $\{\Diamond, \vee\}/\{\Box, \wedge\}$-term and t any term,*
(3) *inequalities $s \preceq t$ with s any term, and t a $\{\Box, \wedge\}/\{\Diamond, \vee\}$-term.*

Example 6.19. As examples of MacNeille canonical equations for residuated lattices we mention the following:

$$\begin{array}{rcl} \Diamond(x_1 \vee x_2) & \approx & \Box(x_1 \wedge x_2) \\ \Box(\Diamond(x_1 \vee x_2) \wedge (\Diamond x_3 \vee x_4)) & \approx & \Box(x_1 \vee x_2) \wedge (x_3 \vee x_4) \\ \Diamond\Box x & \preceq & \Box\Diamond x. \end{array}$$

The first equation is preserved under taking MacNeille completions because both its left and its right hand side are $\{\Diamond, \vee\}/\{\Box, \wedge\}$ terms, and the second, because both sides are $\{\Box, \wedge\}/\{\Diamond, \vee\}$-terms. The inequality is preserved because $\Diamond\Box x$ is a $\{\Diamond, \vee\}/\{\Box, \wedge\}$ term.

Note that the converse inequality, $\Box\Diamond x \preceq \Diamond\Box x$ does not fall under the scope of Theorem 6.18. In fact, it can easily be shown that this inequality is not MacNeille canonical. Consider the Boolean algebra $\mathbb{B}$ of the finite and cofinite sets of natural numbers, interpret $\Diamond$ as the map sending a set $X \subseteq \omega$ to the collection of natural numbers n that are below some $x \in X$, and let $\Box$ be the Boolean dual of $\Diamond$, i.e., $\Box x = \neg\Diamond\neg x$.

As a last example, it is interesting to look at the (finitary) distributive law

$$x \vee (y \wedge z) \;\approx\; (x \vee y) \wedge (x \vee z) \tag{11}$$

from the current perspective. Note that all the function symbols occurring in (11) evaluate to smooth operations. Regardless of that, it follows from Proposition 6.17 that the left hand side $l(x, y, z)$ of (11) is lower stable, while the right hand side $r(x, y, z)$ is upper stable. From this we obtain, for any lattice $\mathbb{L}$, that $l^{\overline{\mathbb{L}}} = (l^{\mathbb{L}})^{\circ} \leq (l^{\mathbb{L}})^{\bullet} = (r^{\mathbb{L}})^{\bullet} = r^{\overline{\mathbb{L}}}$. However, this already follows from the fact that the MacNeille completion of a lattice is again a lattice: the inequality $l \preceq r$ holds for every lattice. Unfortunately, there is nothing we can say about the converse inequality $r \preceq l$ being valid in $\overline{\mathbb{L}}$. And in fact, as is well known, there are distributive lattices of which the MacNeille completion is not distributive.

Remark 6.20. Similar results can be obtained for *binary* operations that are both left- and right residuated, even if these operations are not smooth. Note that the lower MacNeille extension of such an operation is $(\gamma^{\uparrow}, \gamma^{\uparrow})$-continuous. From this it follows that $\{\star, \Diamond, \vee\}/\{\Box, \wedge\}$ terms are lower MacNeille stable (here $\star$ denotes the operation symbol for the doubly residuated binary operations). It is then not hard to derive that, for instance, the associativity and commutativity of $\star$ are lower MacNeille canonical properties. Also, inequalities of the form $s \preceq t$ with s a $\{\star, \Diamond, \vee\}/\{\Box, \wedge\}$ term and t an arbitrary term, are lower MacNeille canonical. It would be interesting to check whether all the results of Ono [37, 39, 38] can be proved via this topological method.

6.5. Uniform lattice expansions.

Finally, we turn to expansions of lattices with operations that are not necessarily order preserving. We restrict our attention to *uniform* operations, and, given our experiences with order preserving operations, we also impose some kind of residuation constraints on the additional operations.

Definition 6.21. Two lattice maps $f : \mathbb{L} \to \mathbb{M}$ and $g : \mathbb{M} \to \mathbb{L}$ are said to form a *Galois connection* if they satisfy, for all $x \in L$ and $y \in M$:

$$y \leq f(x) \text{ iff } x \leq g(y).$$

A *dual Galois connection* is a pair (f, g) of lattice maps such that, for all $x \in L$, $y \in M$ it holds that $y \geq f(x)$ iff $x \geq g(y)$. $\lhd$

It is not difficult to see that the maps $f : \mathbb{L} \to \mathbb{M}$ and $g : \mathbb{M} \to \mathbb{L}$ form a Galois connection iff f, when seen as a lattice map from $\mathbb{L}$ to the order dual $\mathbb{M}^\partial$ of $\mathbb{M}$, is residuated by g, seen as a lattice map between the lattices $\mathbb{M}^\partial$ and $\mathbb{L}$. Likewise, $f : \mathbb{L} \to \mathbb{M}$ and $g : \mathbb{M} \to \mathbb{L}$ form a dual Galois connection iff $f : \mathbb{L}^\partial \to \mathbb{M}$ is residuated by $g : \mathbb{M} \to \mathbb{L}^\partial$. From this the following proposition is an immediate consequence of earlier results on residuated pairs, by the principle of order duality.

Proposition 6.22. *Let f and g form a (dual) Galois connection between the lattices $\mathbb{L}$ and $\mathbb{M}$. Then*

(1) *both f and g are antitone, completely join reversing (meet reversing) lattice maps,*
(2) *both f and g are smooth,*
(3) *$\overline{f}$ and $\overline{g}$ form a (dual) Galois connection between $\overline{\mathbb{L}}$ and $\overline{\mathbb{M}}$,*
(4) *both $\overline{f}$ and $\overline{g}$ are $(\rho^\downarrow, \rho^\uparrow)$ and $(\gamma^\uparrow, \gamma^\downarrow)$-continuous ($(\rho^\uparrow, \rho^\downarrow)$ and $(\gamma^\downarrow, \gamma^\uparrow)$-continuous).*

Let us now study the class of lattice expansions in which the added operations all display some kind of residuation.

Definition 6.23. Let $\mathcal{U}$ be an arbitrary but fixed lattice expansion type that can be divided into four sets $\{\Diamond_i \mid i \in I\}$, $\{\Box_j \mid j \in J\}$, $\{\rhd_k \mid k \in K\}$ and $\{\lhd_l \mid l \in L\}$.

A *uniform residuated lattice expansion (of type $\mathcal{U}$)* is any $\mathcal{U}$-expanded lattice $\mathbb{A}$ in which every diamond is interpreted as a residuated operation, every box as a residual, every symbol $\rhd$ as part of a Galois connection, and every symbol $\lhd$ as part of a dual Galois connection. The class of these algebras is denoted as $\mathsf{URLE}_{\mathcal{U}}$. $\lhd$

When it comes to MacNeille canonicity, the results for the smaller class of residuated lattice expansions can be extended to the class $\mathsf{URLE}_{\mathcal{U}}$. We briefly mention one of the key results here.

Definition 6.24. An occurrence of a variable in a $\mathcal{U}^+$-term is *positive* (*negative*) if it is in the scope of an even (odd, respectively) number of symbols in $\{\rhd, \lhd\}$. An $\mathcal{U}^+$-term t is *uniform* if each variable occurs either only positively or only negatively in t, and *positive* (*negative*) if each variable occurs only positively (negatively, respectively) in t. $\lhd$

Proposition 6.25. *Let t be some uniform $\mathcal{U}^+$-term, and let $\mathbb{A}$ be an algebra in $\mathsf{URLE}_{\mathcal{U}}$. Then*

(1) *both $t^{\mathbb{A}}$ and $t^{\overline{\mathbb{A}}}$ are uniform, and μ-monotone for the same monotonicity type μ,*
(2) *t is lower expanding and upper contracting,*
(3) *t is lower stable if $t^{\overline{\mathbb{A}}}$ is $(\rho, \gamma^\uparrow)$-continuous, and upper stable if $t^{\overline{\mathbb{A}}}$ is $(\rho, \gamma^\downarrow)$-continuous.*

Proof. The first part is proved by a straightforward induction on the complexity of the term t. For part 2 we also use term induction. Omitting the cases that are the same as in the proof of Proposition 6.6, we focus on one of the more interesting cases that involves an order reversing operation, say, when t is of the form $\rhd s$.

Then $(t^{\mathbb{A}})^{\circ} = (\rhd^{\mathbb{A}} \circ s^{\mathbb{A}})^{\circ}$, by definition of term functions. Now $t^{\mathbb{A}}$ is μ-monotone for some monotonicity type μ by the first part of the proposition, so that $t^{\mathbb{A}} : \mathbb{A}^{\mu} \to \mathbb{A}$ is isotone. It is easy to see that s is uniform as well, and that $s^{\mathbb{A}} : \mathbb{A}^{\mu} \to \mathbb{A}$ is *antitone*.

Now let, given a lattice map $f : \mathbb{L} \to \mathbb{M}$, and elements $\epsilon, \eta \in \{1, \partial\}$, $f^{\epsilon,\eta}$ be the lattice map from $\mathbb{L}^{\epsilon}$ to $\mathbb{M}^{\eta}$ that agrees with f as a set-theoretic function. The point is that $t^{\mathbb{A}}$ can now be written as a composition of two *isotone* maps $(s^{\mathbb{A}})^{1,\partial} : \mathbb{A}^{\mu} \to \mathbb{A}^{\partial}$ and $(\rhd^{\mathbb{A}})^{\partial,1} : \mathbb{A}^{\partial} \to \mathbb{A}$.

We may then apply Proposition 5.19, obtaining

$$(t^{\mathbb{A}})^{\circ} \leq \big((\rhd^{\mathbb{A}})^{\partial,1} \circ (s^{\mathbb{A}})^{1,\partial}\big)^{\circ} \leq \big((\rhd^{\mathbb{A}})^{\partial,1}\big)^{\circ} \circ \big((s^{\mathbb{A}})^{1,\partial}\big)^{\circ}$$

Now it is easy to see that for any map f, we have $(f^{1,\partial})^{\circ} = f^{\bullet}$ and $(f^{\partial,1})^{\circ} = f^{\circ}$, so that we find

$$(t^{\mathbb{A}})^{\circ} \leq (\rhd^{\mathbb{A}})^{\circ} \circ (s^{\mathbb{A}})^{\bullet} = \rhd^{\overline{\mathbb{A}}} \circ (s^{\mathbb{A}})^{\bullet}.$$

This gives, by the inductive hypothesis on s and the antitonicity of $\rhd^{\overline{\mathbb{A}}}$, that

$$(t^{\mathbb{A}})^{\circ} \leq \rhd^{\overline{\mathbb{A}}} \circ s^{\overline{\mathbb{A}}},$$

which shows that, indeed, t is lower expanding. For the proof that t is upper contracting we refer to the principle of order duality.

Finally, the last part of the proof is again a completely straightforward generalization of Proposition 6.6 □

On the basis of the Propositions 6.22 and 6.25, many results may be proved concerning the MacNeille canonicity of $\mathcal{U}^{+}$-equations, analogous to the results in the previous subsection. However, it is less straightforward to give a concise, general, formulation of the analogue of Theorem 6.18, see Gehrke, Nagahashi & Venema [23] for the analogous situation in the theory of canonical extensions. Therefore, we omit further details concerning the most general case, and focus on the special case of (residuated expansions of) ortholattices. First however, we discuss one representative example in the general setting of uniform residuated lattice expansions.

Example 6.26. Consider the $\mathcal{U}^{+}$-equation

$$\Diamond(\Box(x_1 \wedge x_2) \vee \rhd x_3) \approx \lhd\Box(x_1 \wedge x_3). \tag{12}$$

For any algebra $\mathbb{A}$ in $\mathsf{URLE}_{\mathcal{U}}$, both the left- and the right hand side term of (12) are interpreted as $(\rho, \gamma^{\uparrow})$-continuous term functions in the algebra $\overline{\mathbb{A}}$. From this, the MacNeille canonicity of (12) is immediate, by Proposition 6.25.

For instance, for the $(\rho, \gamma^{\uparrow})$-continuity of $(\Diamond(\Box(x_1 \wedge x_2) \vee \rhd x_3))^{\overline{\mathbb{A}}}$, first observe that the term function $(\Box(x_1 \wedge x_2))^{\overline{\mathbb{A}}}$ is $(\rho^{\uparrow}, \rho^{\uparrow})$-continuous, while $(\rhd x_3)^{\overline{\mathbb{A}}}$ is $(\rho^{\downarrow}, \rho^{\uparrow})$-continuous. Hence, both term functions are $(\rho, \rho^{\uparrow})$-continuous, and hence, $(\rho, \gamma^{\uparrow})$-continuous. But then the term function associated with the disjunction $\Box(x_1 \wedge x_2) \vee \rhd x_3$ is $(\rho, \gamma^{\uparrow})$-continuous as well. Finally then, using the $(\gamma^{\uparrow}, \gamma^{\uparrow})$-continuity of $\Diamond^{\mathbb{A}}$

we find that indeed, the left hand side of (12) is interpreted as a $(\rho, \gamma^{\uparrow})$-continuous term function.

The last kind of lattice expansion that we consider is that of residuated expansions of ortholattices. We first discuss ortholattices.

Definition 6.27. Let $\mathbb{L} = \langle L, \wedge, \vee, \top, \bot \rangle$ be some lattice. A *complementation* on $\mathbb{L}$ is a map $n : L \to L$ such that $a \vee n(a) = \top$ and $a \wedge n(a) = \bot$ for all $a \in L$. An *orthocomplementation* or *orthonegation* on $\mathbb{L}$ is a complementation n which is both antitone and *idempotent*: $n(n(a)) = a$ for each $a \in L$. An *ortholattice* is an algebra $\mathbb{O} = \langle O, \wedge, \vee, \top, \bot, \neg \rangle$ such that $\neg : O \to O$ is an orthonegation on the lattice $\langle O, \wedge, \vee, \top, \bot \rangle$. $\lhd$

As we already mentioned in the introduction, it is well known that the class of ortholattices, which happens to be a variety, is closed under taking MacNeille completions. Our analysis of this phenomenon is as follows.

Proposition 6.28. *Let $\neg$ be an orthocomplementation on the lattice $\mathbb{L}$. Then*

(1) *$\neg$ forms both a Galois connection and a dual Galois connection with itself,*
(2) *$\neg$ is smooth,*
(3) *$\overline{\neg}$ is $(\rho^{\downarrow}, \rho^{\uparrow})$, $(\rho^{\uparrow}, \rho^{\downarrow})$, $(\gamma^{\downarrow}, \gamma^{\uparrow})$ and $(\gamma^{\uparrow}, \gamma^{\downarrow})$-continuous,*
(4) *$\overline{\neg}$ is an orthocomplementation on $\overline{\mathbb{L}}$.*

Proof. In order to see that $\neg$ forms a Galois connection with itself, assume that $a \leq \neg b$. Then $b = \neg\neg b$ by idempotence, and $\neg\neg b \leq \neg a$ since $\neg$ reverses the order. Taken together, we find that $b \leq \neg a$, as required. We leave it as an exercise for the reader to prove that $(\neg, \neg)$ is a dual Galois connection. Part 2 and 3 are then immediate by Proposition 6.22.

For the last part of the proposition, antitonicity of $\overline{\neg}$ follows from Proposition 6.22. For idempotence of $\overline{\neg}$, first observe that the term function $(\neg\neg x)^{\overline{\mathbb{A}}}$ is $(\gamma^{\uparrow}, \gamma^{\uparrow})$-continuous by part three of this proposition, and hence $(\rho, \gamma^{\uparrow})$-continuous since $\gamma^{\uparrow} \subseteq \rho$. Thus the term $\neg\neg x$ is lower stable by Proposition 6.25, and from this MacNeille canonicity of the equation $\neg\neg x \approx x$ is immediate.

Finally, note that in order to show that $\overline{\neg}$ is a complementation on $\overline{\mathbb{A}}$, we cannot use any of our earlier propositions, since the terms $x \wedge \neg x$ and $x \vee \neg x$ are not uniform. Fortunately, the proof is not very hard. In order to arrive at a contradiction, suppose that for some x in $\overline{\mathbb{A}}$, $x \wedge \overline{\neg} x > \bot$, then by meet density of L, there is some $a \in L$ such that $a \leq x \wedge \overline{\neg} x$ but $a \not\leq \bot$, that is, $a > \bot$. It follows from $a \leq \overline{\neg} x$ that $x \leq \overline{\neg} a = \neg a$, so that $a \leq x \leq \neg a$. From this we may derive that $a \leq a \wedge \neg a = \bot$, which gives the desired contradiction. $\square$

The following is an (almost) immediate corollary.

Corollary 6.29. *The varieties of ortholattices and of Boolean algebras are MacNeille canonical.*

Proof. The result on ortholattices is immediate from part 4 of the previous proposition. For the result on Boolean algebras we need the (easily proved) facts that an ortholattice $\mathbb{O}$ is a Boolean algebra iff the lattice reduct $\mathbb{O}_{\flat}$ is distributive iff the term $\neg x \vee y$ provides a Heyting implication on $\mathbb{O}$. $\square$

Some interesting observations can be made of *expansions* of ortholattices. To start with, we do not need to consider (dual) Galois connections separately, since they can be encoded as residual pairs using the orthocomplementation. The pair (f, g) between $\mathbb{L}$ and $\mathbb{M}$ is a Galois connection iff the composition $\neg^{\mathbb{L}} \circ g : \mathbb{M} \to \mathbb{L}$ is residuated by the map $f \circ \neg^{\mathbb{L}} : \mathbb{L} \to \mathbb{M}$.

Definition 6.30. Let $\mathcal{RO}$ be an arbitrary but fixed lattice expansion type that can be divided into three sets, $\{\neg\}$, $\{\Diamond_i \mid i \in I\}$ and $\{\Box_j \mid j \in J\}$. A *residuated ortholattice expansion (of type $\mathcal{RO}$)* is any $\mathcal{RO}$-expanded lattice $\mathbb{A}$ in which every diamond is interpreted as a residuated operation, every box as a residual, and $\neg$ as an orthocomplementation. $\lhd$

It is not hard to prove an analog of Theorem 6.18 in this setting. As we will see now, we can generalize the Sahlqvist theorem of Givant & Venema to this more general setting.

Definition 6.31. A *left Sahqlvist term* is any term that can be obtained from a $\{\Diamond, \vee\}/\{\Box, \wedge\}$-term by (uniformly) substituting some variables with negative terms. A *Sahlqvist inequality* is an equality of the form $s \preceq t$, with s a left Sahlqvist term, and t a positive term. $\lhd$

Example 6.32. Typical examples of Sahlqvist inequalities are:

$$\begin{array}{rcl} \Diamond(x_1 \wedge \Diamond\neg x_1 \wedge \Box\neg\Box x_2 \wedge x_2) & \preceq & \Box(x_1 \vee x_2) \\ \Diamond\Box x & \preceq & \Box\Diamond x. \end{array}$$

The left hand side of the first inequality can be obtained from the term $\Diamond(x_1 \wedge y_1 \wedge \Box y_2 \wedge x_2)$, by substituting $\Diamond\neg x_1$ for y_1, and $\neg\Box x_2$ for y_2.

A typical counterexample is $\Box\Diamond x \preceq \Diamond\Box x$; in Example 6.19 we already saw that this equation is not preserved under taking MacNeille completions.

Theorem 6.33. *Sahlqvist inequalities are MacNeille canonical over the variety of residuated ortholattice expansions.*

Proof. Consider an arbitrary Sahlqvist inequality $s' \preceq t$. Without loss of generality we may assume that s' is of the form $s^\tau = u(v_1^\tau, \ldots, v_k^\tau)$, such that u is a $\{\vee, \Diamond\}$-term, the v_i are $\{\wedge, \Box\}$-terms, and τ is a substitution replacing each variable x_i $(1 \leq i \leq n)$ with a term of the form $\neg w_i$, with w_i positive. Also without loss of generality we may assume that the x_i do not occur in the w_j, and hence, by definition of τ, do not occur in s^τ either (if this is not the case, then change to a different s and τ). Then it is easy to see that on the class of residuated ortholattice expansions, the inequality $s' \preceq t$ is equivalent to the quasi-equation

$$\left(\mathop{\&}_{1 \leq i \leq n} x_i \preceq \neg w_i \right) \ \Rightarrow \ u(v_1, \ldots, v_k) \preceq t. \tag{13}$$

Here we use the symbol & to denote the conjunction of formulas (equations). The key in proving the equivalence of $s' \leq t$ and (13) is the fact that u will always be interpreted as an order preserving function.

Now suppose that we *add* a binary operation symbol $\bowtie$ to the language, and that we interpret this symbol as the *ortho-order discriminator* on every algebra.

(The idea of expanding the similarity type with such a special operator goes back to Jónsson [30].) That is, we have

$$\bowtie^{\mathbb{A}} (a,b) = \begin{cases} \bot & \text{if } a \leq \neg b, \\ \top & \text{if } a \not\leq \neg b. \end{cases}$$

Then clearly the quasi-equation (13) is equivalent to the formula

$$u(v_1, \dots, v_k) \;\preceq\; t \vee \bigvee_{1 \leq j \leq n} x_j \bowtie w_j. \tag{14}$$

It is then straightforward to check that the left hand side of (14) is a $\{\vee, \Diamond\}/\{\wedge, \Box\}$-term, and the right hand side is positive. Hence, it follows from the results in the previous subsection that (14) is MacNeille-canonical. And since the (unique) MacNeille extension of $\bowtie^{\mathbb{A}}$ is the ortho-order discriminator $\bowtie^{\overline{\mathbb{A}}}$ of $\overline{\mathbb{A}}$, from the fact that (14) holds on $\overline{\mathbb{A}}$ we may deduce that our original inequality $s' \preceq t$ holds on $\overline{\mathbb{A}}$ as well. In other words, $s' \preceq t$ is a MacNeille canonical inequality. $\Box$

Using ideas from Gehrke, Nagahashi & Venema [23], one may formulate more general versions of Theorem 6.33. Alternatively, note that residuated ortholattice expansions provide a very versatile context for the manipulation of equations and inequalities. Many formulas that are not in the proper shape of a Sahlqvist inequality, can be transformed into an equivalent inequality that is covered by either Theorem 6.33 or 6.18, or their order duals.

Example 6.34. Consider for instance the equation $\Box\Diamond(x_1 \vee \neg x_2) \approx \Box(x_1 \vee x_2)$.

To start with, this equation is equivalent to the conjunction of the inequality $\Box\Diamond(x_1 \vee \neg x_2) \preceq \Box(x_1 \vee x_2)$ and its opposite $\Box\Diamond(x_1 \vee \neg x_2) \succeq \Box(x_1 \vee x_2)$.

Using the idempotence of the orthocomplementation it is immediate that the first inequality is equivalent to $\Box\Diamond(\neg y_1 \vee \neg x_2) \preceq \Box(\neg y_1 \vee x_2)$, and hence to $\neg\Box(\neg y_1 \vee x_2) \preceq \neg\Box\Diamond(\neg y_1 \vee \neg x_2)$. Introduce a new diamond $\Diamond' z := \neg\Box\neg z$; it is easy to see that $\Diamond'$ is residuated, and since the last version of the inequality is equivalent to $\Diamond'(y_1 \wedge \neg x_2) \preceq \neg\Box\Diamond(\neg y_1 \vee \neg x_2)$, MacNeille canonicity follows from Theorem 6.33.

MacNeille canonicity of the opposite inequality follows in fact from Theorem 6.33 by the principle of order duality: the term $\Box\Diamond(x_1 \vee \neg x_2)$ can be obtained from $\Box\Diamond(x_1 \vee z_2)$ by the substitution $z_2 \mapsto \neg x_2$, and the term $\Box(x_1 \vee x_2)$ is positive.

7. Further research

There are several problems that seem to be of interest for further investigations. To start with, some questions come up naturally when we compare MacNeille canonicity with preservation results that involve the canonical extension. In the introduction we mentioned already the result of Gehrke, Harding & Venema [19], that every MacNeille canonical variety of uniform lattice expansions is canonical (i.e., closed under taking canonical extensions).

(1) Given a finite, uniform, lattice expansion, when does it generate a MacNeille canonical variety? In particular, is this a decidable problem?

Gehrke & Harding [18] proved that *every* such algebra generates a canonical variety. In the case of MacNeille completions this is not true, as is witnessed by the variety of distributive lattices. Baker's theorem [2] may be of use here, providing finite axiomatizations for finitely generated varieties of lattice expansions. However, since in general it is not decidable whether a given equation is MacNeille canonical, a straightforward application of this result will not suffice.

(2) Can MacNeille canonicity of a variety always be proved using the syntactic shape of the equations defining the variety?

In the case of the canonical extension, there are 'semantic' results as well. In particular, a result in modal logic, originating with Fine [16], states that if K is an *elementary* class of relational structures, its associated class of complex algebras generates a canonical variety, see Goldblatt, Hodkinson & Venema [25] for further discussion. It would be interesting to have such results for MacNeille completions as well, and to investigate whether these results can be derived from the syntactic proofs in this paper.

(3) Is every MacNeille canonical variety of lattice expansions generated by some elementary class of relational structures?

This question is related to another result of Gehrke, Harding & Venema [19], and to the previous question. In the case of Boolean algebras with operators, the cited paper proves that every MacNeille canonical variety is not just canonical, but elementarily generated (that is: generated by the complex algebras of some elementary class of relational structures). The problem is whether this result carries over to other kinds of lattice expansions; obviously, this problem involves the duality theory for lattices and their expansions, see for instance [13].

And finally, we mention two more specific problems for further research.

(4) First, it would we interesting to undertake a more systematic study of residuated lattices (that is, lattices expanded with a residuated *binary* operation), perhaps with additional structure. Apart from Remark 6.20, we did not discuss this in any kind of detail.

(5) Second, an intrigueing technical question concerning modal algebras is whether lower MacNeille canonicity implies upper MacNeille canonicity.

Of the above questions, the first and last originate with G. Bezhanishvili and J. Harding.

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