Introduction to Modal Logic (Fall 2007)

Seventh (and final) homework assignment

- Deadline: December 21, 09.00 hrs. This deadline is completely strict.
  Hand in your homework by one of the following two methods:
  ∘ put it in Yde Venema's pigeon hole in Euclides
  ∘ send it as a pdf file to both Yde Venema (yde@science.uva.nl) and Jacob Vosmaer (jvosmaer@science.uva.nl).
- Please make **exactly three exercises** from the ones below.
- Your choice must include **at least one of the exercises 5 and 6**.
- Make sure that each exercise is on a **separate** piece of paper, which contains your name.
- Grading is from 0 to 100 points; you get 10 points for free.
- Success!

`30 pt` **Exercise 1 (finite model property and filtration)**
Call a frame or model *euclidean* if it satisfies $\forall xyz\,((Rxy \land Rxz) \to Ryz)$, and let E be the class of euclidean models. Fix a formula ξ, and let Σ be the smallest subformula closed set of formulas containing ξ that satisfies, for all formulas ψ: if $\Diamond\psi \in \Sigma$, then $\Box\Diamond\psi \in \Sigma$. (Recall that $\Box$ is an abbreviation of $\neg\Diamond\neg$.) Note that in general, Σ will be infinite.

(a) Prove that $\mathsf{E} \Vdash \Diamond\psi \to \Box\Diamond\psi$.

(b) Prove that every euclidean model can be filtrated through Σ to a euclidean model.

(c) Every euclidean model satisfies the following modal reduction principles: $\Diamond\Diamond\Diamond \leftrightarrow \Diamond\Diamond$, $\Diamond\Diamond\Box \leftrightarrow \Diamond\Box$, $\Diamond\Box\Diamond \leftrightarrow \Diamond\Diamond$ and $\Diamond\Box\Box \leftrightarrow \Diamond\Box$. Prove this for the third principle; that is, prove that all formulas of the form $\Diamond\Box\Diamond\varphi \leftrightarrow \Diamond\Diamond\varphi$, are true throughout every euclidean model.

(d) Prove that the basic modal similarity type has the finite model property with respect to the class of euclidean models.

`30 pt` **Exercise 2 (ultrafilter extensions)**
Let B and B^ω denote the sets of finite and infinite strings of 0s and 1s, respectively. (Note that infinite strings can also be seen as maps from $\mathbb{N}$ to $\{0,1\}$.) We let $\sqsubset$ denote the (proper) initial segment relation; that is, $s \sqsubset t$ holds if s is a proper initial segment of t.
Let $\mathbb{B}$ denote the frame $\mathbb{B} = (B, \sqsubset)$, and let $\mathbb{B}'$ denote the frame $\mathbb{B}' = (B \cup B^\omega, R)$, with Rst if either s is finite and $s \sqsubset t$ or s is infinite and $s = t$. (Equivalently, R is the relation $\sqsubset$ made reflexive on infinite strings).

(a) Prove that every nonprincipal ultrafilter u over B uniquely determines an infinite string $\sigma_u \in B^\omega$ which is defined by the following property: $\pi_s \sqsubset^{ue} u$ for each $s \in B$ such that $s \sqsubset \sigma_u$ (i.e. $\forall s \in B.[s \sqsubset \sigma_u \ \rightarrow \pi_s \sqsubset^{ue} u]$).

(b) Extend this map to a bounded morphism from $\mathfrak{ue}\mathbb{B}$ onto $\mathbb{B}'$.

`30 pt` **Exercise 3 (Grzegorczyk's formula)**
Prove that Grzegorczyk's formula

$$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$$

characterizes the class of frames (W, R) such that

1. R is reflexive,
2. R is transitive, and
3. there are no infinite paths $x_0 R x_1 R x_2 R \dots$ such that for all $i \in \mathbb{N}$ we have $x_i \neq x_{i+1}$.

`30 pt` **Exercise 4 (General frames and normal modal logics)**
Let K be a class of general frames. Prove that the set Λ_{K} is closed under uniform substitution. (Hint: given a valuation V and a substitution σ, consider the valuation V_σ given by $V_\sigma(p) := V(\sigma(p))$, where $\sigma(p)$ is the formula that replaces p according to σ.)

`30 pt` **Exercise 5 (Semantically driven completeness)**
Let $\mathcal{F} = (\mathbb{N}, S)$ the frame consisting of the set of natural numbers together with the successor relation, i.e. for all $m, n \in \mathbb{N}$ we have Smn if $n = m + 1$. Axiomatize the logic of $\mathcal{F}$ and show that your axiomatization is complete. (Hint: One extra axiom should be enough.)

`30 pt` **Exercise 6 (Finitary methods in completeness)**
Consider a modal similarity type with three diamonds $\Diamond_1, \Diamond_2$ and $\Diamond_C$.[1]
We define the logic L to be the smallest normal modal logic that contains:

(4_i)	$\Diamond_i \Diamond_i p \rightarrow \Diamond_i p$	for $i = 1, 2$
(T_i)	$p \rightarrow \Diamond_i p$	for $i = 1, 2$
(B_i)	$p \rightarrow \Box_i \Diamond_i p$	for $i = 1, 2$
$(\Diamond_C)$	$\Diamond_C p \leftrightarrow (p \vee \Diamond_1 \Diamond_C p \vee \Diamond_2 \Diamond_C p)$	
(Ind)	$\Box_C(p \rightarrow (\Box_1 p \wedge \Box_2 p)) \rightarrow (p \rightarrow \Box_C p)$	

Furthermore we call a frame (W, R_1, R_2, R_C) *standard* if $R_i \subseteq W \times W$, $i = 1, 2$, is an equivalence relation and if

$$R_C = (R_1 \cup R_2)^*,$$

where $(\_)^*$ denotes the reflexive, transitive closure of a relation.

[1] The intended interpretation of these operators is *epistemic*: in terms of the corresponding universal modalities: $\Box_i \varphi$, for $i = 1, 2$ encodes that agent i *knows* φ, while $\Box_C \varphi$ means that φ is *common knowledge* between the two agents, i.e. $\Box_C \varphi$ can be seen as an abbreviation of the formula $\varphi \wedge \Box_i \varphi \wedge \Box_2 \varphi \wedge \Box_1 \Box_2 \varphi \wedge \dots$.

Prove that L is complete with respect to the class of finite, standard frames!

You can follow the following steps (analogue to the completeness proof of PDL in Section 4.8):

- Define an analogue of the Fisher-Ladner closure for the logic L.
- Define a finite canonical model $(At(\Sigma), R_1, R_2, S_C, V)$ for L. The relations R_i on the finite canonical model should be defined as follows:

$$AR_iB \quad \text{if for all } \Diamond_i\varphi \in \neg FL(\Sigma) \qquad \Diamond_i\varphi \in A \text{ iff } \Diamond_i\varphi \in B.$$

- Prove that $S_C \subseteq R_C := (R_1 \cup R_2)^*$.
- Turn the finite canonical model into a standard one and continue as in the completeness proof for PDL.

You do not need to write down in details the parts of the proof that are completely analogous to that of the completeness proof for PDL.