Immediate consequences

It follows from the definitions that the set $\mu_{MLD}$ is closed under taking negations. Informally, let $\sim \varphi$ be the result of simultaneously replacing all occurrences of $\top$ with $\bot$, of $p$ with $\neg p$ (for free variables $p$), of $\land$ with $\lor$, of $\Box_d$ with $\Diamond_d$, of $\mu x$ with $\nu x$, and vice versa, while leaving occurrences of bound variables unchanged. As an example, $\sim (\mu x.p \land \Diamond x) = \nu x.\neg p \land \Box x$.

Formally, it is easiest to define $\sim \varphi$ via the Boolean dual of $\varphi$.

**Definition 3.22** Given a modal fixpoint formula $\varphi$, we define its Boolean dual $\varphi^\theta$ inductively as follows:

<table>
<thead>
<tr>
<th>$\bot^\theta$</th>
<th>$\top^\theta$</th>
<th>$p^\theta$</th>
<th>$(\varphi \lor \psi)^\theta$</th>
<th>$(\varphi \land \psi)^\theta$</th>
<th>$(\Box_d \varphi)^\theta$</th>
<th>$(\Diamond_d \varphi)^\theta$</th>
<th>$(\mu x. \varphi)^\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bot$</td>
<td>$\top$</td>
<td>$p$</td>
<td>$\varphi \lor \psi$</td>
<td>$\varphi \land \psi$</td>
<td>$\Box_d \varphi$</td>
<td>$\Diamond_d \varphi$</td>
<td>$\mu x. \varphi$</td>
</tr>
<tr>
<td>$(\neg p)^\theta$</td>
<td>$\neg p$</td>
<td></td>
<td>$\varphi$</td>
<td>$\varphi$</td>
<td>$\Box_d \varphi$</td>
<td>$\Diamond_d \varphi$</td>
<td>$\mu x. \varphi$</td>
</tr>
<tr>
<td>$(\varphi \land \psi)^\theta$</td>
<td>$(\varphi \lor \psi)^\theta$</td>
<td></td>
<td>$p$</td>
<td>$p$</td>
<td>$\Box_d \varphi$</td>
<td>$\Diamond_d \varphi$</td>
<td>$\mu x. \varphi$</td>
</tr>
</tbody>
</table>

Based on this definition, we define the formula $\sim \varphi$ as the formula $\varphi^\theta[p \mapsto p \land \Box x]$, that we obtain from $\varphi^\theta$ by replacing all occurrences of $p$ with $\neg p$, and vice versa, for all free proposition letters $p \in FV(\varphi)$.

The following proposition states that $\sim$ functions as a standard Boolean negation. We let $\sim S X := S \setminus X$ denote the complement of $X$ in $S$.

**Proposition 3.23** Let $\varphi$ be a modal fixpoint formula. Then $\sim \varphi$ corresponds to the negation of $\varphi$, that is,

$$\llbracket \sim \varphi \rrbracket^S = \sim S [\llbracket \varphi \rrbracket^S]$$

for every labelled transition system $S$.

**Proof.** We first show, by induction on $\varphi$, that $\varphi^\theta$ corresponds to the Boolean dual of $\varphi$. For this purpose, given a labelled transition system $S = (S, R, V)$, we let $S^\sim$ denote the complemented model, that is, the structure $(S, R, V^\sim)$, where $V^\sim(p) := \sim S V(p)$. Then we claim that

$$\llbracket \varphi^\theta \rrbracket^S = \sim S [\llbracket \varphi \rrbracket^{S^\sim}]$$

and we prove this statement by induction on the complexity of $\varphi$. Leaving all other cases as exercises for the reader, we concentrate on the inductive case where $\varphi$ is of the form $\mu x. \psi$. Then we may show that, for an arbitrary subset $U \subseteq S$:

$$\llbracket (\psi^\theta)^{\mu x}_x \rrbracket^S(U) = \llbracket \psi^\theta \rrbracket^{S[U]} = \sim S [\llbracket \psi \rrbracket^{(S[U]^\sim)}] = \sim S [\llbracket \psi \rrbracket^{(S [x \mapsto \sim S U])}] = (\psi^{S^\sim})^\theta(U),$$

where we use the inductive hypothesis on $\psi$ and $S[U]$ in the second equality. Clearly this implies that

$$(\psi^\theta)^{\mu x}_x = (\psi^{S^\sim})^\theta.$$
We now turn to the proof of (11) for the case where $\varphi = \mu x. \psi$:

\[
\begin{align*}
[(\mu x. \psi)^\varnothing]^S &= [\nu x. \psi^\varnothing]^S \\
&= \text{GFP}(\psi^\varnothing)^S_x \\
&= \text{GFP}(\psi^S)^\varnothing \\
&= \sim_s \text{LFP.}\psi^S_x \\
&= \sim_s [\mu x. \psi]^S
\end{align*}
\] (Definition $(\mu x. \psi)^\varnothing$

(Theorem 3.21)

(Equation (12))

(Proposition 3.11)

(Theorem 3.21)

To obtain (10) from (11), first observe that we have

\[
\left[\chi[p := \neg p \mid p \in FV(\chi)]\right]^S = \left[\chi\right]^S
\] (13)

for any formula $\chi$. But then, taking $\varphi^\varnothing$ for $\chi$, we find that

\[
\left[\sim \varphi\right]^S = \left[\varphi^\varnothing\right]^S = \sim_s [\varphi]^S = \sim_s [\varphi]^S,
\]

where the first equality holds by (13) and the definition of $\sim \varphi$, the second equality is (11), and the third equality follows from the trivial observation that $(S^S)^\varnothing = S$.

QED

**Remark 3.24** It follows from the Proposition above that we could indeed have based the language of the modal $\mu$-calculus on a smaller alphabet of primitive symbols. Given a set $D$ of atomic actions, we could have defined the set of modal fixpoint formulas using the following induction:

\[
\begin{align*}
\varphi ::= & T \mid p \mid \neg \varphi \mid \varphi \lor \varphi \mid \square_d \varphi \mid \mu x. \varphi
\end{align*}
\]

where $p$ and $x$ are propositional variables, $d \in D$, and in $\mu x. \varphi$, all free occurrences of $x$ must be positive (that is, under an even number of negation symbols). Here we define $FV(\neg \varphi) = FV(\varphi)$ and $BV(\neg \varphi) = BV(\varphi)$.

In this set-up, the connectives $\land$ and $\square_q$ are defined using the standard abbreviations, while for the greatest fixpoint operator we may put

\[
\mu x. \varphi := \neg \mu x. \neg \varphi(\neg x).
\]

Note the *triple* use of the negation symbol here — which can be explained by thinking of $\neg \varphi(\neg x)$ as the formulas $\varphi^\varnothing$.

Earlier on we defined the notions of *clean* and *guarded* formulas.

**Proposition 3.25** *Every fixpoint formula is equivalent to a clean one.*

**Proof.** We leave this proof as an exercise for the reader. QED

**Proposition 3.26** *Every fixpoint formula is equivalent to a guarded one.*
Proof. (Sketch) We prove this proposition by formula induction. Clearly the only nontrivial case to consider concerns the fixpoint operators. Consider a formula of the form $\eta x.\delta(x)$, where $\delta(x)$ is guarded and clean, and suppose that $x$ has an unguarded occurrence in $\delta$.

First consider an unguarded occurrence of $x$ in $\delta(x)$ inside a fixpoint subformula, say, of the form $\theta y.\gamma(x, y)$. By induction hypothesis, all occurrences of $y$ in $\gamma(x, y)$ are guarded. Obtain the formula $\delta'$ from $\delta$ by replacing the subformula $\theta y.\gamma(x, y)$ with $\gamma(x, \theta y.\gamma(x, y))$. Then clearly $\delta'$ is equivalent to $\delta$, and all of the unguarded occurrences of $x$ in $\delta$ are outside of the scope of the fixpoint operator $\theta$.

Continuing like this we obtain a formula $\eta x.\delta(x)$ which is equivalent to $\eta x.\delta(x)$, and in which none of the unguarded occurrences of $x$ lies inside the scope of a fixpoint operator. That leaves $\land$ and $\lor$ as the only operation symbols in the scope of which we may find unguarded occurrences of $x$.

From now on we only consider the case where $\eta = \mu$, leaving the very similar case where $\eta = \nu$ as an exercise. Clearly, using the laws of classical propositional logic, we may bring the formula $\delta$ into conjunctive normal form

$$(x \lor \alpha_1(x)) \land \cdots \land (x \lor \alpha_n(x)) \land \beta(x),$$

where all occurrences of $x$ in $\alpha_1, \ldots, \alpha_n$ and $\beta$ are guarded. (Note that we may have $\beta = \top$, or $\alpha_i = \bot$ for some $i$.)

Clearly (14) is equivalent to the formula

$$\delta'(x) := (x \lor \alpha(x)) \land \beta(x),$$

where $\alpha = \alpha_1 \land \cdots \land \alpha_n$. Thus we are done if we can show that

$$\mu x.\delta'(x) \equiv \mu x.\alpha(x) \land \beta(x).$$

(15)

Since $\alpha \land \beta$ implies $\delta'$, it is easy to see (and left for the reader to prove) that $\mu x.\alpha(x) \land \beta$ implies $\mu x.\delta'$. For the converse, it suffices to show that $\varphi := \mu x.\alpha(x) \land \beta(x)$ is a prefixpoint of $\delta'(x)$. But it is not hard to derive from $\varphi \equiv \alpha(\varphi) \land \beta(\varphi)$ that

$$\delta'(\varphi) = (\varphi \lor \alpha(\varphi)) \land \beta(\varphi) \equiv ((\alpha(\varphi) \land \beta(\varphi)) \lor \alpha(\varphi)) \land \beta(\varphi) \equiv \alpha(\varphi) \land \beta(\varphi) \equiv \varphi,$$

which shows that $\varphi$ is in fact a fixpoint, and hence certainly a prefixpoint, of $\delta'(x)$. QED

**Remark 3.27** a note on size matters: what is the size of the smallest guarded equivalent of $\varphi$?

Combining the proofs of the previous two propositions one easily shows the following.

**Proposition 3.28** Every fixpoint formula is equivalent to a clean, guarded one.

### 3.5 Adequacy

In this section we prove the equivalence of the two semantic approaches towards the modal $\mu$-calculus. Since the algebraic semantics is usually taken to be the more fundamental notion, we refer to this result as the Adequacy Theorem stating, informally, that games are an adequate way of working with the algebraic semantics.
Theorem 3.29 (Adequacy) Let $\xi$ be a clean $\mu ML_D$-formula. Then for all labelled transition systems $\mathcal{S}$ and all states $s$ in $\mathcal{S}$:

$$s \in [\xi]^{\mathcal{S}} \iff (\xi, s) \in \text{Win}_\exists(\mathcal{E}(\xi, \mathcal{S})).$$

Proof. The theorem is proved by induction on the complexity of $\xi$. We only discuss the inductive step where $\xi$ is of the form $\eta x.\delta$, leaving the other cases as exercises to the reader. Our proof for this inductive case is based on two key observations, concerning, respectively, the role of the unfolding game for $\delta_2^S$ in the semantics for $\mu x.\delta$, and the similarity between the evaluation games for $\xi$ and for $\delta$.

Starting with the first, note that in the cases that we are dealing with, the set $[\eta x.\delta]^S$ is the least/greatest fixed point of the map $\delta_2^S: \wp(S) \to \wp(S)$. By our earlier Theorem 3.14 on unfolding games, it holds that

$$[\eta x.\delta]^S = \text{Win}_\exists(\mathcal{U}^0(\delta_2^S)).$$

Hence, in order to prove (16), it suffices to show that, for any state $s_0$:

$$s_0 \in \text{Win}_\exists(\mathcal{U}^0(\delta_2^S)) \iff (\xi, s_0) \in \text{Win}(\mathcal{E}(\xi, \mathcal{S})).$$

(18)

In other words, the crucial tasks in the proof of this inductive step concern the transformation of a winning strategy for $\exists$ in the unfolding game $\mathcal{U}^0(\delta_2^S)@s_0$ to a winning strategy for her in the evaluation game $\mathcal{E}(\xi, \mathcal{S}@)(\xi, s_0)$, and vice versa.

If we then look at the unfolding game for $\delta_2^S$ in a bit more detail, we note that a round of this game, starting at position $s \in S$, consists of $\exists$ picking a subset $A \subseteq S$ that is subject to the constraint that $s \in \delta_2^S(A) = [\delta]^S[x \mapsto A]$. But by the inductive hypothesis we have, for all $A \subseteq S$,

$$s \in \delta_2^S(A) \iff (\delta, s) \in \text{Win}_\exists(\mathcal{E}(\delta, \mathcal{S}[x \mapsto A])).$$

(19)

The importance of (19) brings us to the comparison between the games $\mathcal{G} := \mathcal{E}(\xi, \mathcal{S})$ and $\mathcal{G}_A := \mathcal{E}(\delta, \mathcal{S}[x \mapsto A])$. The second key observation in the inductive step for the fixpoint operators is that these games are very similar indeed. For a start, the positions of the two games are essentially the same. Positions of the form $(\xi, t)$, which exist in the first game but not in the second, are the only exception — but in $\mathcal{G}$, any position $(\xi, t)$ is immediately and automatically succeeded by the position $(\delta, t)$ which does exist in the second game. What is important is that the positions for $\exists$ are exactly the same in the two games, and thus we may apply her positional strategies for the one game in the other game as well. The only real difference between the games shows up in the rule concerning positions of the form $(x, u)$. In $\mathcal{G}_A$, $x$ is a free variable ($x \in \text{FV}(\delta)$), so in a position $(x, u)$ the game is over, the winner being determined by $u$ being a member of $A$ or not. In $\mathcal{G}$ however, $x$ is bound, so in position $(x, u)$, the variable $x$ will get unfolded to $\delta$.

Combining these two observations, the key insight in the proof of (18) will be to think of $\mathcal{E}(\xi, \mathcal{S})$ as a variant of the unfolding game $\mathcal{U} := \mathcal{U}^0(\delta_2^S)$ where each round of $\mathcal{U}$ corresponds to a version of the game $\mathcal{G}_T$, with $T$ being the subset of $S$ lastly picked by $\exists$ in $\mathcal{U}$. We are now ready for the details of the proof of (18).
For the direction from left to right of (18), suppose that $\exists$ has a winning strategy in the game $\mathcal{U}$ starting at some position $s_0$. Without loss of generality (see Exercise 3.6) we may assume that this strategy is positional, so we may represent it as a map $T : S \to \wp(S)$. By the legitimacy of this strategy, for every $s \in \text{Win}_\exists(\mathcal{U})$ it holds that $s \in \delta^2_x(T_s)$. So by the inductive hypothesis (19), for each such $s$ we may assume the existence of a winning strategy $f_s$ for $\exists$ in the game $G_{T_s}(\delta, s)$. Given the similarities between the games $\mathcal{G}$ and $G_{T_s}$ (see the discussion above), this strategy is also applicable in the game $\mathcal{G} @ (\delta, s)$, at least, until a new position of the form $(x, t)$ is reached.

This suggests the following strategy $g$ for $\exists$ in $\mathcal{G} @ (\xi, s_0)$:

1. after the initial automatic move, the position of the match is $(\delta, s_0)$; $\exists$ first plays her strategy $f_{s_0}$;
2. each time a position $(x, s)$ is reached, the match automatically moves to position $(\delta, s)$; distinguish cases:
   (a) if $s \in \text{Win}_\exists(\mathcal{U})$ then $\exists$ continues with $f_s$;
   (b) if $s \notin \text{Win}_\exists(\mathcal{U})$ then $\exists$ continues with a random strategy.

First we show that this strategy guarantees that whenever a position of the form $(x, s)$ is visited, $s$ belongs to $\text{Win}_\exists(\mathcal{U})$, so that case (b) mentioned above never occurs. The proof is by induction on the number of positions $(x, s)$ that have been visited already. For the inductive step, if $s$ is a winning position for $\exists$ in $\mathcal{U}$, then, as we saw, $f_s$ is a winning strategy for $\exists$ in the game $G_{T_s}(\delta, s)$. This means that if a position of the form $(x, t)$ is reached, the variable $x$ must be $\text{true}$ at $t$ in the model $S[x \rightarrow T_s]$, and so $t$ must belong to the set $T_s$. But by assumption of the map $T : S \to \wp(S)$ being a winning strategy in $U$, any element of $T_s$ is again a member of $\text{Win}_\exists(\mathcal{U})$.

In fact we have shown that every unfolding of the variable $x$ in $\mathcal{G}$ marks a new round in the unfolding game $\mathcal{U}$. To see why the strategy $g$ guarantees a win for $\exists$ in $\mathcal{G} @ (\xi, s_0)$, consider an arbitrary $\mathcal{G} @ (\xi, s_0)$-match $\pi$ in which $\exists$ plays $g$. Distinguish cases.

First suppose that $x$ is unfolded only finitely often. Let $(x, s)$ be the last basic position in $\pi$ where this happens. Given the similarities between the games $\mathcal{G}$ and $G_{T_s}$, the match from this moment on can be seen as both a $g$-guided $\mathcal{G}$-match and an $f_s$-guided $G_{T_s}$-match. As we saw, $f_s$ is a winning strategy for $\exists$ in the game $G_{T_s}(\delta, s)$. But since no further position of the form $(x, t)$ is reached, $\mathcal{G}$ and $G_{T_s}$ only differ when it comes to $x$, this means that $\pi$ is also a win for $\exists$ in $\mathcal{G}$.

If $x$ is unfolded infinitely often during the match $\pi$, then by the fact that $\xi = \eta x.\delta$, it is the highest variable that is unfolded infinitely often. We have to distinguish the case where $\eta = \nu$ from that where $\eta = \mu$. In the first case, $\exists$ is the winner of the match $\pi$, and we are done. If $\eta = \mu$, however, $x$ is a least fixpoint variable, and so $\exists$ would lose the match $\pi$. We therefore have to show that this situation cannot occur. Suppose for contradiction that $s_1, s_2, \ldots$ are the positions where $x$ is unfolded. Then it is easy to verify that the sequence $s_0T_{s_0}s_1T_{s_1}\ldots$ constitutes a $U^\mu$-match in which $\exists$ plays her strategy $T$. But this is not possible, since $T$ was assumed to be a winning strategy for $\exists$ in the least fixpoint game $\mathcal{U} = U^\mu(\delta^2_\exists)$. 


Lectures on the modal $\mu$-calculus
For the converse implication of (18), we will show how each of \( \exists \)'s positional winning strategies \( f \) in \( \mathcal{G} \) induces a positional strategy \( U_f \) for her in \( \mathcal{U} \), and that this strategy \( U_f \) is winning for her starting at every position \( s \in W := \{ s \in S \mid (\xi, s) \in \text{Win}_\exists(\mathcal{G}) \} \).

Fix a positional winning strategy \( f \) for \( \exists \) in \( \mathcal{G} \); that is, \( \exists \) is guaranteed to win any \( f \)-guided match starting at a position \( (\varphi, t) \in \text{Win}_\exists(\mathcal{G}) \). Observe that, as discussed above, we may and will treat \( f \) as a positional strategy in each of the games \( \mathcal{G}_A \) as well.

Given a state \( s \in W \), we let \( \mathcal{T}_f(s) \) be the strategy tree induced by \( f \) in \( \mathcal{G}_A @ (\delta, s) \); that is, for some unspecified set \( A \subseteq S \), the nodes of \( \mathcal{T}_f \) consist of all \( f \)-guided matches in \( \mathcal{G}_A \) that start at \( (\delta, s) \). Thus by definition, any such \( \Sigma \) satisfies first(\( \Sigma \)) = \( (\delta, s) \). To define the successor relation of \( \mathcal{T}_f \), let \( \Sigma \) be an arbitrary \( f \)-guided match. If last(\( \Sigma \)) is a position owned by \( \exists \), then \( \Sigma \) will have a single successor in \( \mathcal{T}_f \), viz., the unique extension of \( \Sigma \) with the position \( f(\Sigma) \) picked by \( f \). On the other hand, if last(\( \Sigma \)) is owned by \( \forall \), then any possible continuation \( \Sigma \cdot b \), where \( b \) is an admissible position picked by \( \forall \), is a successor of \( \Sigma \). Finally, we let \( U_f(s) \) be the set of states \( u \) such that the position \( (x, u) \) occurs somewhere in \( \mathcal{T}_f \). It is easy to see that any \( \mathcal{G}_A \)-match \( \Sigma \) ending in the position of the form \( (x, u) \), is finished immediately, and thus provides a leaf of the tree \( \mathcal{T}_f \). It is also an easy consequence of the definitions that, whenever \( t \in U_f(s) \) for some \( s \in W \), then there is an \( f \)-guided match \( \Sigma_{s,t} \) such that first(\( \Sigma_{s,t} \)) = \( (\delta, s) \) and last(\( \Sigma_{s,t} \)) = \( (x, t) \). Note that this match \( \Sigma_{s,t} \) can be seen both as a (full) \( \mathcal{G}_A \)-match and as a (partial) \( \mathcal{G} \)-match.

Viewing \( U_f \) as a positional strategy for \( \exists \) in \( \mathcal{U} \), we claim that in fact it is a winning strategy for her in \( \mathcal{U} @ s_0 \). Before proving this, we state and prove two auxiliary claims on \( U_f \). First we observe that

\[
\text{if } s \in W \text{ then } s \in \delta_2(U_f(s)). \tag{20}
\]

For a proof of (20), it is obvious from the definition of \( U_f(s) \) that \( f \) is a positional winning strategy for \( \exists \) in \( \mathcal{G}_{U_f(s)} = \mathcal{E}(\delta, \exists[s \to U_f(s)]) \) starting at \( (\delta, s) \). But then by the inductive hypothesis on \( \delta \) we obtain that \( \exists[s \to U_f(s)], s \models \delta \), or, equivalently, \( s \in \delta_2(U_f(s)) \).

Second, we claim that

\[
\text{if } s \in W \text{ then } U_f(s) \subseteq W. \tag{21}
\]

To see this, first note that if \( s \in W \) then by definition \( (\xi, s) \in \text{Win}_\exists(\mathcal{G}) \); but from this it is immediate that \( (\delta, s) \in \text{Win}_\exists(\mathcal{G}) \), and since we assumed \( f \) to be a positional winning strategy for \( \exists \) in \( \mathcal{G} \), it follows by definition of \( U_f(s) \) that for every \( u \in U_f(s) \) the position \( (x, u) \) is winning for \( \exists \) in \( \text{Win}_\exists(\mathcal{G}) \). But from this it is easy to derive that both \( (\delta, s) \) and \( (\xi, s) \) are winning position for \( \exists \) in \( \mathcal{G} \) as well. The latter fact then shows that \( u \in W \) and since \( u \) was an arbitrary element of \( U_f(s) \), (21) follows.

We can now prove that \( U_f \) is a winning strategy for \( \exists \) in \( \mathcal{U} @ s_0 \). First of all, it follows from (20) that \( U_f(s) \) is a legitimate move in \( \mathcal{U} \) for every position \( s \in W \). From this and (21) we may conclude that \( \exists \) never gets stuck in an \( U_f \)-guided \( \mathcal{U} \)-match starting at \( s_0 \); that is, she wins every finite \( U_f \)-guided \( \mathcal{U} \)-match. In case \( \eta = \nu \) this suffices, since in \( UC^\nu(\delta_2^\nu) \) all infinite matches are won by \( \exists \).

Where \( \eta = \mu \) we have a bit more work to do, since in this case all infinite matches of \( UC^\mu(\delta_2^\nu) \) are won by \( \forall \). Suppose for contradiction that \( \Sigma = s_0 U_f(s_0) s_1 U_f(s_1) \cdots \) would be an infinite \( U_f \)-guided match of \( UC^\mu(\delta_2^\nu) \). Then for every \( i \in \omega \) we have that \( s_{i+1} \in U_f(s_i) \), so that there is a partial \( f \)-guided match \( \Sigma_i = \Sigma_{s_i,s_{i+1}} \) with first(\( \Sigma_i \)) = \( (\delta, s_i) \) and last(\( \Sigma_i \)) = \( (x, s_{i+1}) \). But
then it is straightforward to verify that the infinite match \( \Sigma_G := \Sigma_0 \cdot \Sigma_1 \cdot \Sigma_2 \cdots \) we obtain by concatenating the individual \( f \)-guided matches \( \Sigma_i \), constitutes an infinite \( f \)-guided \( G \)-match with \( \text{first}(\Sigma_G) = \text{first}(\Sigma_0) = (\xi, s_0) \). Since the highest fixpoint variable unfolded infinitely often during \( \Sigma_G \) obviously would be \( x \), this match would be lost by \( \exists \). Here we arrive at the desired contradiction, since \((\xi, s_0) \in \text{Win}_\exists(G)\), and \( f \) was assumed to be a positional winning strategy in \( G \).

QED

**Convention 3.30** In the sequel we will use the Adequacy Theorem without further notice. Also, we will write \( S, s \models \varphi \) in case \( s \in \llbracket \varphi \rrbracket^S \).

**Notes**

What we now call the Knaster-Tarski Theorem (Theorem 3.4) was first proved by Knaster [12] in the context of power set algebras, and subsequently generalized by Tarski [29] to the setting of complete lattices. The Bekić principle (Proposition 3.16) stems from an unpublished technical report.

▷ more references to be supplied

As far as we know, the results in section 3.2 on the duality between the least and the greatest fixpoint of a monotone map on a complete Boolean algebra, are folklore. The characterization of least and greatest fixpoints in game-theoretic terms is fairly standard in the theory of (co-)inductive definitions, see for instance Aczel [1]. The equivalence of the algebraic and the game-theoretic semantics of the modal \( \mu \)-calculus (here formulated as the Adequacy Theorem 3.29) was first established by Emerson & Jutla [9].

**Exercises**

**Exercise 3.1** Prove Proposition 3.6: show that monotone maps on complete lattices are inductive.

**Exercise 3.2** Prove Theorem 3.21.

(Hint: given complete lattices \( C \) and \( D \), and a monotone map \( f : C \times D \to C \), show that the map \( g : D \to C \) given by

\[
g(d) := \mu x. f(x, d)
\]

is monotone. Here \( f(x, d) \) is the least fixpoint of the map \( f_d : C \to C \) given by \( f_d(c) = f(c, d) \).)

**Exercise 3.3** Let \( F : \wp(S) \to \wp(S) \) be some monotone map. A collection \( D \in \wp(S) \) of subsets of \( S \) is directed if for every two sets \( D_0, D_1 \in D \), there is a set \( D \in D \) with \( D_i \subseteq D \) for \( i = 0, 1 \). Call \( F \) (Scott) continuous if it preserves directed unions, that is, if \( F(\bigcup D) = \bigcup_{D \in D} F(D) \) for every directed \( D \).

Prove the following:

(a) \( F \) is Scott continuous iff for all \( X \subseteq S \): \( F(X) = \bigcup\{ F(Y) \mid Y \subseteq X \} \).

(Here \( Y \subseteq X \) means that \( Y \) is a finite subset of \( X \).)
(b) If $F$ is Scott continuous then the unfolding ordinal of $F$ is at most $\omega$.

(c) Give an example of a Kripke frame $S = (S, R)$ such that the operation $[R]$ is not continuous.

(d) Give an example of a Kripke frame $S = (S, R)$ such that the operation $[R]$ has closing/unfolding ordinal $\omega + 1$.

**Exercise 3.4** By a mutual induction we define, for every finite set $P$ of propositional variables, the fragment $\mu\text{ML}_p^C$ by the following grammar:

$$\varphi ::= p | \psi \lor \varphi | \varphi \land \varphi | \Diamond \varphi | \mu q.\varphi',$$

where $p \in P$, $\psi \in \mu\text{ML}$ is a $P$-free formula, and $\varphi' \in \mu\text{ML}_{p,\varphi(q)}^C$.

Prove that for every Kripke model $S$, every formula $\varphi \in \mu\text{ML}_p^C$, and every proposition letter $p \in P$, the map $\varphi^S_p: \varphi(S) \rightarrow \varphi(S)$ is continuous.

**Exercise 3.5** Let $F: \varphi(S) \rightarrow \varphi(S)$ be a monotone operation, and let $\gamma_F$ be its unfolding ordinal. Sharpen Corollary 3.7 by proving that the cardinality of $\gamma_F$ is bounded by $|S|$ (rather than by $|\varphi(S)|$).

**Exercise 3.6** Prove that the unfolding game of Definition 3.12 satisfies positional determinacy. That is, let $U^\mu(F)$ be the least fixpoint unfolding game for some monotone map $F: \varphi(S) \rightarrow \varphi(S)$. Prove the existence of two positional strategies $f_\exists: S \rightarrow \varphi(S)$ and $f_\forall: \varphi(S) \rightarrow S$ such that for every position $p$ of the game, either $f_\exists$ is a winning strategy for $\exists$ in $U^\mu(F)@p$, or else $f_\forall$ is a winning strategy for $\forall$ in $U^\mu(F)@p$. 