5 Infinite board games

Much of the work linking (fixpoint) logic to automata theory involves nontrivial concepts and results from the theory of infinite games. In this chapter we discuss some of the highlights of this theory in a fair amount of detail. This allows us to be rather informal about game-theoretic concepts in the rest of the notes.

5.1 Board games

The games that we are dealing with here can be classified as board or graph games. They are played by two agents, here to be called 0 and 1.

Definition 5.1 If $\Pi \in \{0, 1\}$ is a player, then $\Pi$ denotes the opponent $1 - \Pi$ of $\Pi$.

A board game is played on a board or arena, which is nothing but a directed graph in which each node is marked with either 0 or 1. A match of the game consists of the two players moving a pebble or token across the board, following the edges of the graph. To regulate this, the collection of graph nodes, usually referred to as positions of the game, is partitioned into two sets, one for each player. Thus with each position we may associate a unique player whose turn it is to move when the token lies on position $p$.

Definition 5.2 A board is a structure $B = (B_0, B_1, E)$, such that $B_0$ and $B_1$ are disjoint sets, and $E \subseteq B^2$, where $B := B_0 \cup B_1$. We will make use of the notation $E[p]$ for the set of admissible or legitimate moves from a board position $p \in B$, that is, $E[p] := \{q \in B \mid (p, q) \in E\}$. Positions not in $E[p]$ will sometimes be referred to as illegitimate moves with respect to $p$. A position $p \in B$ is a dead end if $E(p) = \emptyset$. If $p \in B$, we let $\Pi_p$ denote the (unique) player such that $p \in B_{\Pi_p}$, and say that $p$ belongs to $\Pi_p$, or that it is $\Pi_p$’s turn to move at $p$.

A match of the game may in fact be identified with the sequence of positions visited during play, and thus corresponds to a path through the graph.

Definition 5.3 A path through a board $B = (B_0, B_1, E)$ is a (finite or infinite) sequence $\Sigma \in B^\infty$ such that $E_\Sigma \Sigma_{i+1}$ whenever applicable. A full or complete match through $B$ is either an infinite $B$-path, or a finite $B$-path $\Sigma$ ending with a dead end (i.e. $E[\text{last}(\Sigma)] = \emptyset$).

A partial match is a finite path through $B$ that is not a match; in other words, the last position of a partial match is not a dead end. We let $\text{PM}_{\Pi}$ denote the set of partial matches such that $\Pi$ is the player whose turn it is to move at the last position of the match. In the sequel, we will denote this player as $\Pi_\Sigma$; that is, $\Pi_\Sigma := \Pi_{\text{last}(\Sigma)}$.

Each full or completed match is won by one of the players, and lost by their opponent; that is, there are no draws. A finite match ends if one of the players gets stuck, that is, is forced to move the token from a position without successors. Such a finite, completed, match is lost by the player who got stuck.

The importance of this explains the definition of the notion of a subboard. Note that any set of positions on a board naturally induces a board of its own, based on the restricted edge relation. We will only call this structure a subboard, however, if there is no disagreement between the two boards when it comes to players being stuck or not.
Definition 5.4 Given a board $B = \langle B_0, B_1, E \rangle$, a subset $A \subseteq B$ determines the following board $B_A := (A \cap B_0, A \cap B_1, E_{|A})$, where $E_{|A} := E \cap (A \times A)$ is the restriction of $E$ to $A$. This structure is called a subboard of $B$ if for all $p \in A$ it holds that $E[p] = \emptyset$ iff $E_{|A}[p] = \emptyset$. \hfill \triangledown

If neither player ever gets stuck, an infinite match arises. The flavor of a board game is very much determined by the winning conditions of these infinite matches.

Definition 5.5 Given a board $B$, a winning condition is a map $W : B^\omega \to \{0, 1\}$. An infinite match $\Sigma$ is won by $W(\Sigma)$. A board game is a structure $G = \langle B_0, B_1, E, W \rangle$ such that $(B_0, B_1, E)$ is a board, and $W$ is a winning condition on $B$. \hfill \triangledown

Although the winning condition given above applies to all infinite $B$-sequences, it will only make sense when applied to matches. We have chosen the above definition because it is usually much easier to formulate maps that are defined on all sequences.

Before players can actually start playing a game, they need a starting position. The following definition introduces some terminology and notation.

Definition 5.6 An initialized board game is a pair consisting of a board game $G$ and a position $q$ on the board of the game; such a pair is usually denoted $G@q$.

Given a (partial) match $\Sigma$, its first element $\text{first}(\Sigma)$ is called the starting position of the match. We let $\text{PM}_\Pi(q)$ denote the set of partial matches for $\Pi$ that start at position $q$. \hfill \triangledown

Central in the theory of games is the notion of a strategy. Roughly, a strategy for a player is a method that the player uses to decide how to continue partial matches when it is their turn to move. More precisely, a strategy is a function mapping partial plays for the player to new positions. It is a matter of definition whether one requires a strategy to always assign moves that are legitimate, or not; here we will not make this requirement.

Definition 5.7 Given a board game $G = \langle B_0, B_1, E, W \rangle$ and a player $\Pi$, a $\Pi$-strategy, or a strategy for $\Pi$, is a map $f : \text{PM}_\Pi \to B$. In case we are dealing with an initialized game $G@q$, then we may take a strategy to be a map $f : \text{PM}_\Pi(q) \to B$. A match $\Sigma$ is consistent with or guided by a $\Pi$-strategy $f$ if for any $\Sigma' \subseteq \Sigma$ with $\text{last}(\Sigma') \in B_\Pi$, the next position on $\Sigma$ (after $\Sigma'$) is indeed the element $f(\Sigma')$.

A $\Pi$-strategy $f$ is surviving in $G@q$ if the moves that it prescribes to $f$-guided partial matches in $\text{PM}_\Pi@p$ are always admissible to $\Pi$, and winning for $\Pi$ in $G@p$ if in addition all $f$-guided full matches starting at $p$ are won by $\Pi$. A position $q \in B$ is winning for $\Pi$ if $\Pi$ has a winning strategy for the game $G@q$; the collection of winning positions for $\Pi$ in $G$ is denoted as $\text{Win}_\Pi(G)$. \hfill \triangledown

Intuitively, $f$ being a surviving strategy in $G@q$ means that $\Pi$ never gets stuck in an $f$-guided match of $G@q$, and so guarantees that $\Pi$ can stay in the game forever.

Convention 5.8 In practice, it will often be convenient to extend the definition of a strategy to include maps $f$ that are partial in the sense that they are only defined on a proper subset
of PM$_\Pi$. We will only permit ourselves such a sloppiness if we can guarantee that $f(\Sigma)$ is defined for every $\Sigma \in$ PM$_\Pi$ that is consistent with the partial $\Pi$-strategy $f$, so that the situation where the partial strategy actually would fail to suggest a move, will never occur.

**Definition 5.9** The game $G$ on the board $B$ is determined if $\text{Win}_0(G) \cup \text{Win}_1(G) = B$; that is, each position is winning for one of the players.

In principle, when deciding how to move in a match of a board game, players may use information about the entire history of the match played thus far. However, it will turn out to be advantageous to work with strategies that are simple to compute. This applies for instance to so-called finite-memory strategies, which can be computed using only a finite amount of information about the history of the match.

**Definition 5.10** A strategy $f$ is positional or history-free if $f(\Sigma) = f(\Sigma')$ for any $\Sigma, \Sigma'$ with $\text{last}(\Sigma) = \text{last}(\Sigma')$.

**Convention 5.11** A positional $\Pi$-strategy may be represented as a map $f : B_{\Pi} \to B$.

### 5.2 Winning conditions

In case we are dealing with a finite board $B$, then we may nicely formulate winning conditions in terms of the set of positions that occur infinitely often in a given match. But in the case of an infinite board, there may be matches in which no position occurs infinitely often (or more than once, for that matter). Nevertheless, we may still define winning conditions in terms of objects that occur infinitely often, if we make use of finite colorings of the board. If we assign to each position $b \in B$ a color, taken from a finite set $C$ of colors, then we may formulate winning conditions in terms of the colors that occur infinitely often in the match.

**Definition 5.12** A coloring of $B$ is a function $\Gamma$ assigning to each position $p \in B$ a color $\Gamma(b)$ taken from some finite set $C$ of colors. Such a coloring $\Gamma : B \to C$ naturally extends to a map $\Gamma : B^\omega \to C^\omega$ by putting $\Gamma(p_0 p_1 \ldots) := \Gamma(p_0) \Gamma(p_1) \ldots$.

Now if $\Gamma : B \to C$ is a coloring, for any infinite sequence $\Sigma \in B^\omega$, the map $\Gamma \circ \Sigma \in C^\omega$ forms the associated sequence of colors. But then since $C$ is finite there must be some elements of $C$ that occur infinitely often in this stream.

**Definition 5.13** Let $B$ be a board and $\Gamma : B \to C$ a coloring of $B$. Given an infinite sequence $\Sigma \in B^\omega$, we let $\text{Inf}_\Gamma(\Sigma)$ denote the set of colors that occur infinitely often in the sequence $\Gamma \circ \Sigma$. 

\[\text{Inf}_\Gamma(\Sigma) = \{c \in C \mid c \text{ occurs infinitely often in } \Gamma(\Sigma)\}\]
A Muller condition is a collection $\mathcal{M} \subseteq \mathcal{P}(C)$ of subsets of $C$. The corresponding winning condition is defined as the following map $W_{\mathcal{M}} : B^\omega \rightarrow \{0,1\}$:

$$W_{\mathcal{M}}(\Sigma) := \begin{cases} 0 & \text{if } \text{Inf}_{\mathcal{M}}(\Sigma) \in \mathcal{M} \\ 1 & \text{otherwise.} \end{cases}$$

A Muller game is a board game of which the winning conditions are specified by a Muller condition.

In words, player 0 wins an infinite match $\Sigma = p_0 p_1 \ldots$ if the set of colors one meets infinitely often on this path, belongs to the Muller collection $\mathcal{M}$.

>- Examples to be supplied.

Muller games have two nice properties. First, they are determined. This follows from a well-known general game-theoretic result, but can also be proved directly. In addition, we may assume that the winning strategies of each player in a Muller game are finite-memory strategies.

>- Details to be supplied

The latter property becomes even nicer if the Muller condition allows a formulation in terms of a parity map. In this case, as colors we take natural numbers. Note that by definition of a coloring, the range $\Omega[B]$ of the coloring function $\Omega$ is finite. This means that every subset of $\Omega[B]$ has a maximal element. Hence, every match determines a unique natural number, namely, the ‘maximal color’ that one meets infinitely often during the match. Now a parity winning condition states that the winner of an infinite match is 0 if this number is even, and 1 if it is odd. More succinctly, we formulate the following definition.

**Definition 5.14** Let $B$ be some set; a parity map on $B$ is a coloring $\Omega : B \rightarrow \omega$, that is, a map of finite range. A parity game is a board game $G = \langle B_0, B_1, E, W_B \rangle$ in which the winning condition is given by

$$W_B(\Sigma) := \max(\text{Inf}_B(\Sigma)) \mod 2.$$ 

Such a parity game is usually denoted as $G = \langle B_0, B_1, E, \Omega \rangle$.

The key property that makes parity games so interesting is that they enjoy positional determinacy. We will prove this in section 5.4. First we turn to a special case, viz., the reachability games.

### 5.3 Reachability Games

Reachability games are a special kind of board games. They are played on a board such as described in section 5.1, but now we also choose a subset $A \subseteq B$. The aim of the game is for the one player to move the pebble into $A$ and for the other to avoid this to happen.

**Definition 5.15** Fix a board $\mathcal{B}$ and a subset $A \subseteq B$. The reachability game $R(\mathcal{B}, A)$ is then defined as the game over $\mathcal{B}$ in which $\Pi$ wins as soon as a position in $A$ is reached or if $\Pi$ gets stuck. On the other hand, $\Pi$ wins if he can manage to keep the token outside of $A$ infinitely long, or if $\Pi$ gets stuck.
Remark 5.16 If we want reachability games to fit the format of a board game exactly, we have to modify the board, as follows. Given a reachability game $R_{\Pi}(B,A)$, define the board $B' := \langle B'_0, B'_1, E' \rangle$ by putting:

$$
\begin{align*}
B'_0 &:= B_0 \setminus A \\
B'_1 &:= B_1 \cup A \\
E' &:= \{(p,q) \in E \mid p \not\in B_\Pi \cap A\}.
\end{align*}
$$

In other words, $B'$ is like $B$ except that player $\Pi$ gets stuck in a position belonging to $A$. Furthermore, the winning conditions of such a game are very simple: simply define $W : B^\omega \to \{0,1\}$ as the constant function mapping all infinite matches to $\Pi$. This can be formulated as a parity condition: put $\Omega(p) := \Pi$, for every $p \in B$.

Since reachability games can thus be formulated as very simple parity games, the following theorem, stating that reachability games enjoy positional determinacy, can be seen as a warming up exercise for the general case.

**Theorem 5.17 (Positional determinacy of reachability games)** For any reachability game $R$ there are positional strategies $f_0$ and $f_1$ for 0 and 1, respectively, such that for every position $q$ there is a player $\Pi$ such that $f_\Pi$ is a winning strategy for $\Pi$ in $R@q$.

**Proof.** We leave this proof as an exercise to the reader. $\text{QED}$

**Definition 5.18** The winning region for $\Pi$ in $R_{\Pi}(B,A)$ is called the attractor set of $\Pi$ for $A$ in $B$, notation: $\text{Attr}_{\Pi}(B,A)$. In the sequel we will fix a positional winning strategy for $\Pi$ in $R_{\Pi}(B,A)$ and denote it as $\text{attr}_{\Pi}(B,A)$.

Note that $\Pi$-attractor sets always contain all points from which $\Pi$ can make sure that $\Pi$ gets stuck. Furthermore, it is easy to see that in $\text{attr}_{\Pi}(A)$-guided matches the pebble never leaves $\text{Attr}_{\Pi}(A)$ (at least if the match starts inside $\text{Attr}_{\Pi}(A)$!)

**Proposition 5.19** $\text{Attr}_{\Pi}$ is a closure operation on $\mathcal{P}(B)$, i.e.

1. $A \subseteq A'$ implies $\text{Attr}_{\Pi}(A) \subseteq \text{Attr}_{\Pi}(A')$,
2. $A \subseteq \text{Attr}_{\Pi}(A)$,
3. $\text{Attr}_{\Pi} (\text{Attr}_{\Pi}(A)) = \text{Attr}_{\Pi}(A)$.

A kind of counterpart to attractor sets are $\Pi$-traps. In words, a set $A$ is a $\Pi$-trap if $\Pi$ can’t get the pebble out of $A$, while her opponent has the power to keep it inside $A$.

**Definition 5.20** Given a board $B$, we call a subset $A \subseteq B$ a $\Pi$-trap if $E[b] \subseteq A$ for all $b \in A \cap B_{\Pi}$, while $E[b] \cap A \neq \emptyset$ for all $b \in A \cap B_{\Pi}$.

Note that a $\Pi$-trap does not contain $\Pi$-endpoints and that $\Pi$ will therefore never get stuck in a $\Pi$-trap. We conclude this section with a useful proposition.
Proposition 5.21 Let $\mathbb{B}$ be a board and $A \subseteq B$ an arbitrary subset of $B$. Then the following assertions hold.

1. If $A$ is a $\Pi$-trap then $A$ is a subboard of $B$.
2. The union $\bigcup \{A_i \mid i \in I\}$ of an arbitrary collection of $\Pi$-traps is again a $\Pi$-trap.
3. If $A$ is a $\Pi$-trap then so is $\text{Attr}_\Pi(A)$.
4. The complement of $\text{Attr}_\Pi(A)$ is a $\Pi$-trap.
5. If $A$ is a $\Pi$-trap in $B$ then any $C \subseteq A$ is a $\Pi$-trap in $\mathbb{B}_A$.

Proof. All statements are easily verified and thus the proof is left to the reader. \[\text{QED}\]

5.4 Positional Determinacy of Parity Games

Theorem 5.22 (Positional Determinacy of Parity Games) For any parity game $G$ there are positional strategies $f_0$ and $f_1$ for 0 and 1, respectively, such that for every position $q$ there is a player $\Pi$ such that $f_\Pi$ is a winning strategy for $\Pi$ in $G[q]$.

We start with the definition of players’ paradises. In words, a subset $A \subseteq B$ is a $\Pi$-paradise if $\Pi$ has a positional strategy $f$ guarantees her both that she wins the game, and that the token stays in $A$.

Definition 5.23 Given a parity game $G(\mathbb{B}, \Omega)$, we call a $\Pi$-trap $A$ a $\Pi$-paradise if there exists a positional winning strategy $f : A \cap B_\Pi \to A$.

The following proposition establishes some basic facts about paradises.

Proposition 5.24 Let $G(\mathbb{B}, \Omega)$ be a parity game. Then the following assertions hold:

1. The union $\bigcup \{P_i \mid i \in I\}$ of an arbitrary set of $\Pi$-paradises is again a $\Pi$-paradise.
2. There exists a largest $\Pi$-paradise.
3. If $P$ is a $\Pi$-paradise then so is $\text{Attr}_\Pi(P)$.

Proof. The main point of the proof of part (1) is that we somehow have to uniformly choose a strategy on the intersection of paradises, such that we will end up following the strategy of only one paradise. For this purpose, we assume that we have a well-ordering on the index set $I$ (i.e., for the general case we assume the Axiom of Choice).

For the details, assume that $\{P_i \mid i \in I\}$ is a family of paradises, and let $f_i$ be the positional winning strategy for $P_i$. Note that $P := \bigcup \{P_i \mid i \in I\}$ is a trap for $\Pi$ by Proposition 5.21. Assume that $<$ is a well-ordering of $I$, so that for each $q \in P$ there is a minimal index $\min(q)$ such that $p \in P_{\min(q)}$. Define a positional strategy on $P$ by putting

$$f(q) := f_{\min(q)}(q).$$
This strategy ensures at all times that the pebble either stays in the current paradise, or else it moves to a paradise of lower index, and so, any match where $\Pi$ plays according to $f$ will proceed through a sequence of $\Pi$-paradises of decreasing index. Because of the well-ordering, this decreasing sequence of paradises cannot be strictly decreasing, and thus we know that after finitely many steps the pebble will remain in the paradise where it is, say, $P_j$. From that moment on, the match is continued as an $f_j$-guided match inside $P_j$, and since $f_j$ is by assumption a winning strategy when played inside $P_j$, this match is won by $\Pi$.

Part (2) of the proposition should now be obvious: clearly the union of all $\Pi$-paradises is the greatest $\Pi$-paradise.

In order to prove part (3) we need to show that there exists a winning strategy for $\Pi$. The principal idea is to first move to $-\paradise$. The greatest assumption a winning strategy when played inside $\Pi$ in that moment on, the match is continued as an $\Pi$-paradise according to $f$, else it moves to a paradise of lower index, and so, any match where $\Pi$ plays according to an associated positional strategy after finitely many steps the pebble will remain in the paradise where it is, say, $\Pi_j$. Because of the well-ordering, the pebble proceeds through a sequence of $\Pi$-paradises of decreasing index. From this decreasing sequence of paradises cannot be strictly decreasing, and thus we know that $\Pi$-paradises of decreasing index. Because of the well-ordering, $\Pi$ proceeds through a sequence of this play is won by $\Pi$. However if we start outside $P$ we will at first follow the strategy $\Pi$-paradise. By maximality of $\Pi$-paradise. By proposition 5.24(3) it follows that $\Pi$-trap. By proposition 5.24(3) it follows that $\Pi$-trap and $\Pi$-paradise is a $\Pi$-trap in $\Pi$-paradise is a $\Pi$-trap. By proposition 5.24(3) it follows that $\Pi$-trap is a $\Pi$-trap.

Consider $G_X$, the subgame of $G$ restricted to $X$. Note that by proposition 5.21(1), $X$ is a subboard of $B$, so the name 'subgame' is justified. Define $N := \{b \in X \mid \Omega(b) = n\}$ to be the set of all points in $X$ with priority $n$ and let $Z := X \setminus \Attr\Pi_X B \setminus \Attr\Pi_X (N)$. Since $Z$ is the complement of a $\Pi$-attractor set in $B_X$ it is a $\Pi$-trap in $B_X$ and hence a $\Pi$-trap of $B$, and so, a subboard of $B$.
By the induction hypothesis we can split the subgame $G_Z$ into a 0-paradise $Z_0$ and a 1-paradise $Z_1$, see the picture. The winning strategies in these paradises we call $f_0$ and $f_1$ respectively. (All notions are with regard to the game $G_Z$.) We want to show that $Z_{\Pi} = \emptyset$, so that $Z = Z_{\Pi}$.

To this aim, we claim that $P_{\Pi} \cup Z_{\Pi}$ is a $\Pi$-paradise in $G$, and in order to prove this, we consider the following strategy $g$ of $\Pi$:

\[
g(b) := \begin{cases} 
  f(b) & \text{if } b \in P_{\Pi} \\
  f_{\Pi}(b) & \text{if } b \in Z_{\Pi}.
\end{cases}
\]

It is left as an exercise for the reader to show that this is indeed a winning strategy for $\Pi$ which keeps the pebble inside $P_{\Pi} \cup Z_{\Pi}$. (Here we need the fact that $B_Z$ is a subboard of $B$ — if this were not the case, then we could not rule out the existence of positions that are dead ends for $\Pi$ in $B_Z$, but not in $B$.) By maximality of $P_{\Pi}$ we see that $P_{\Pi} = P_{\Pi} \cup Z_{\Pi}$ and since $P_{\Pi}$ and $Z_{\Pi}$ are disjoint we conclude that $Z_{\Pi}$ is empty indeed.

This means we can write

\[X = Z_{\Pi} \cup Attr_{\Pi}(N).\]

We are now almost ready to define the winning strategy for $\Pi$ which keeps the token inside $X$. Recall that $X$ is a $\Pi$-trap, so that for each $b \in X \cap B_{\Pi}$, we may pick an arbitrary element $k(b) \in E[b] \cap X$. Now define the following strategy $h$ in $G$ for $\Pi$ on $X$.

\[
h(b) := \begin{cases} 
  k(b) & \text{if } b \in N \\
  attr_{\Pi}(N)(b) & \text{if } b \in Attr_{\Pi}(N) \setminus N \\
  f_{\Pi}(b) & \text{if } b \in Z_{\Pi} = Z.
\end{cases}
\]

It is left as an exercise for the reader to show that $h$ is indeed a winning strategy for $\Pi$ in $G$ and that it keeps the pebble in $X$.

Finally, the assertion made in Theorem 5.22 follows directly from this proposition because by definition of paradises there now exists for every point $b \in B$ a positional winning strategy for the game $G(B, \Omega)$.

- strategies as 1-player games
- automatic moves
5.5 Game equivalences

5.6 Size issues and algorithmic aspects

Notes

The application of game-theoretic methods in the area of logic and automata theory goes back to work of Büchi. The positional determinacy of parity games was proved independently by Emerson & Jutla [9] and Mostowski in an unpublished technical report. Our proof of this result is based on Zielonka [33].