Topics in Modal Logic (Fall 2024)

Seventh tutorial (12 December 2024): exercise sheet

Exercise 1 (bisimulation invariance) Let (Y, m) and (Y', m') be one-step models; we call (Y', m') a *quotient* of (Y, m) if there is a *homomorphism* from (Y, m) onto (Y', m'), i.e., a surjection $f : Y \to Y'$ such that $m'(f(y)) = m(y)$, for all $y \in Y$. A one-step formula α is *quotient invariant* if we have $(Y, m) \Vdash^1 \alpha$ iff $(Y', m') \Vdash^1 \alpha$, whenever (Y', m') is a quotient of (Y, m).

Prove that a one-step formula α is quotient invariant iff it is (one-step) bisimulation invariant, cf. Definition 10.32).

Exercise 2 (MSO-automata) Let L be an ω-regular language over the alphabet $\wp(\mathsf{P})$.

(a) Show that there is an MSO-automaton $\mathbb{A}$ over the set P of proposition letters, such that $\mathbb{A}$ accepts a tree model (S, R, V) with root w_0 iff there is an infinite branch[1] $w_0 w_1 w_2 ...$ of the tree, for which the $\wp(P)$-stream $m(w_0)m(w_1)m(w_2)\cdots$ belongs to L (where m is the marking corresponding to V).

(b) Show that there is an MSO-automaton $\mathbb{A}'$ over the set P of proposition letters, such that $\mathbb{A}'$ accepts a tree model (S, R, V) iff, *for at least two infinite branches* of the tree, the associated $\wp(P)$-stream belongs to L.

Hint. Obviously, if $\alpha = (s_n)_{n\in\omega}$ and $\beta = (t_n)_{n\in\omega}$ are distinct branches, there must be a unique 'splitting point', i.e., the first $n \in \omega$ such that $s_n \neq t_n$. Note that this n cannot be zero, since $s_0 = t_0$ is the root of the tree. For the definition of the automaton $\mathbb{A}'$, modify the automaton $\mathbb{A}$, copying some of its states into 'pre' and 'post' splitting states.

A detailed proof of the correctness of your solution is not needed, but you need to provide and motivate the definition of your automata.

Exercise 3 (MSO-automata)

(a) Prove that MSO-automata are closed under union. (That is, show that if K_0 and K_1 are classes of pointed Kripke models that are recognizable by an MSO-automaton, then so is $\mathsf{K}_0 \cup \mathsf{K}_1$.)

(b) Recall that $\mathtt{1FOE}(A)$ denotes the set of monadic first-order sentences that are positive in each $a \in A$. Show that this set is closed under taking *Boolean duals*, in the sense that for each $\alpha \in \mathtt{1FOE}(A)$ there is a sentence $\alpha^\partial \in \mathtt{1FOE}(A)$ with the property that for all domains D and interpretations $V : A \to \wp D$ we have

$$D, V \models \alpha^\partial \text{ iff } D, V^\neg \not\models \alpha,$$

where $V^\neg$ is the complemented valuation given by $V^\neg(a) := D \setminus V(a)$.

[1] Branches of a tree start at the root.

(c) Prove that MSO-automata are closed under complementation. (Hint: given a priority map $\Omega : A \to \omega$, consider the map $\Omega^+ : A \to \omega$ given by $\Omega^+(a) := \Omega(a) + 1$.)