2.8 Disjunctive formulas

In the theory of the modal μ-calculus, a fundamental role is played by the so-called *disjunctive formulas*. These are built using the cover modality discussed in Section 1.7, and, as discussed there in the setting of basic modal logic, characterised by a severely restricted use of conjunctions.

▶ `For the time being we confine attention to the monomodal case`

We first introduce the full language of the nabla-based version of the modal μ-calculus. This is simply the extension of the language $\mathtt{ML}_\nabla$ with fixpoint operators. Recall that in this language we work with the *finitary* versions of conjunction and disjunction.

Definition 2.63 The formulas of the language $\mu\mathtt{ML}_\nabla$ are given by the following grammar:

$$\varphi ::= p \mid \overline{p} \mid \bigvee\Phi \mid \bigwedge\Phi \mid \nabla\Phi \mid \mu x\,\varphi \mid \nu x\,\varphi$$

where p and x are propositional variables, $\Phi \subseteq_\omega \mu\mathtt{ML}_\nabla$, and the formation of the formulas $\eta x\,\varphi$ is subject to the proviso that there are no occurrences of the literal $\overline{x}$ in φ. ◁

As in the basic (fixpoint-free) case, the only conjunctions that we allow in a disjunctive formula are of the form $\alpha \bullet \Phi$. The latter formula stands for the conjunction $(\bigwedge \alpha) \wedge \nabla\Phi$, where α and Φ are finite sets of, respectively, literals and disjunctive formulas. Based on our discussion in the fixpoint-free case, one might suggest to define disjunctive formulas by means of the following grammar:

$$\varphi ::= \top \mid \bigvee\Phi \mid \alpha \bullet \Phi \mid \mu x\,\varphi \mid \nu x\,\varphi,$$

where α is an arbitrary conjunction of literals, and we have the usual positivity constraint on the formation of fixpoint formulas $\eta x\,\varphi$. The question is, however, whether we should allow *bound* variables to occur to the left of a bullet conjunction: such variables represent their unfoldings, and since these unfoldings may be much more complex than literals, if we allow subformulas of the form, say, $\{x, y\} \bullet \nabla\Phi$, effectively we are letting in nontrivial conjunctions through the back door.

For this reason we will make an a priori distinction between free and bound variables, the idea being that the free variables can only occur (positively or negatively) among the literals that occur to the left of the bullet conjunctions, while the bound variables can occur anywhere else but not there. It also turns out that we do not need to take $\top$ as a primitive disjunctive formulas, see Example 2.65 below.

Definition 2.64 Let P be a finite set of propositional variables. To define the set $\mu\mathtt{DML}(\mathsf{P})$ of *(monomodal) disjunctive formulas in* P we start with the expressions given by the following grammar:

$$\varphi ::= x \mid \bigvee\Phi \mid \alpha \bullet \Phi \mid \mu x\,\varphi \mid \nu x\,\varphi$$

where x is a propositional variable not in P, Φ is a finite set of formulas from this grammar, and α is a finite set of literals over P; furthermore, the formulas $\mu x\,\varphi$ and $\nu x\,\varphi$ can only be formed if $\overline{x}$ does not occur in φ.

The set $\mu\mathtt{DML}(\mathsf{P})$ of *disjunctive formulas over* P consists of all expressions ξ that meet this pattern and satisfy the condition that $FV(\xi) \subseteq \mathsf{P}$. We let $\mu\mathtt{DML}$ be the set of formulas that are disjunctive in some set P, and we call a formula *disjunctive* if it belongs to this set $\mu\mathtt{DML}$. $\lhd$

▶ `Note that disjunctive formulas are tidy.`

In practice we will often pretend that atomic formulas, and in fact all propositional formulas in disjunctive normal form, are disjunctive. This can be justified as follows.

Example 2.65 As in the basic case, the constant $\bot$ can be seen as an abbreviation of the disjunctive formula $\bigvee \varnothing$. Different from the basic case, however, we can do without the constant $\top$ as a primitive constant either, since the presence of the greatest fixpoint operator enables us to write

$$\top \equiv \nu x\, (\varnothing \bullet \varnothing \vee \varnothing \bullet \{x\}).$$

A different way of representing the boolean constants as disjunctive formulas is via $\top \equiv \mu x.x$ and $\top \equiv \nu x.x$.

Literals do not qualify as disjunctive formulas, but any literal ℓ is equivalent to a disjunctive formula as well:

$$\ell \equiv \{\ell\} \bullet \{\top\} \vee \{\ell\} \bullet \varnothing.$$

Slightly extending this example, we may represent any finite conjunction of literals by a disjunctive formula as follows:

$$\bigwedge \alpha \equiv \alpha \bullet \{\top\} \vee \alpha \bullet \varnothing.$$

Finally based on this we may in practice pretend that any propositional formula in disjunctive normal form is disjunctive in the current sense as well:

$$\bigvee \big\{ \textstyle\bigwedge \alpha_i \mid i \in I \big\} \equiv \bigvee \big\{ \alpha_i \bullet \{\top\} \vee \alpha_i \bullet \varnothing \mid i \in I \big\}.$$

Another example of a disjunctive formulas is $\mu x \Big(\{p, \overline{q}\} \bullet \big\{ x, \nu y \left(\{p\} \bullet \{x \vee y\} \right) \big\} \Big)$, but not its subformula $\{p, \overline{q}\} \bullet \big\{ x, \nu y \left(\{p\} \bullet \{x \vee y\} \right) \big\}$ (since in the latter formula x is free, and hence, it may not occur in the set to the right of either of the bullet conjunctions). A final example of a non-disjunctive formula is $\mu x \Big(\{\overline{p}, \overline{q}\} \bullet \big\{ x, \nu y \left(\{p, x\} \bullet \{x, \top\} \right) \big\} \Big)$ (here the subformula $\{p, x\} \bullet \{x, \top\}$ is not admissible since x, being a bound variable, may not occur in the set to the left of the bullet conjunction). $\lhd$

Turning to the semantics of disjunctive formulas, below we introduce the *evaluation game* for this language. For this definition we recall that a relation $Z \subseteq S \times S'$ is *full* on some pair $(U, U') \in \wp(S) \times \wp(S')$ if $U \subseteq \mathsf{Dom}(Z)$ and $U' \subseteq Ran(Z)$, or, in other words, if every $u \in U$ is related by Z to some $u' \in U'$, and vice versa.

Definition 2.66 The positions and admissible moves of the evaluation game for clean disjunctive formulas are given in Table 11. The winning conditions are as in the evaluation games for arbitrary $\mu\mathtt{ML}$-formulas. $\lhd$

Position		Player	Admissible moves
$(\bigvee \Phi, s)$		$\exists$	$\{(\varphi, s) \mid \varphi \in \Phi\}$
$(\alpha \bullet \Phi, s)$		$\forall$	$\{(\ell, s) \mid \ell \in \alpha\} \cup \{(\nabla\Phi, s)\}$
(p, s)	with $p \in FV(\xi)$ and $s \in V(p)$	$\forall$	$\varnothing$
(p, s)	with $p \in FV(\xi)$ and $s \notin V(p)$	$\exists$	$\varnothing$
$(\overline{p}, s)$	with $p \in FV(\xi)$ and $s \in V(p)$	$\exists$	$\varnothing$
$(\overline{p}, s)$	with $p \in FV(\xi)$ and $s \notin V(p)$	$\forall$	$\varnothing$
$(\nabla\Phi, s)$		$\exists$	$\{Z \subseteq \Phi \times R[s] \mid Z$ is full on Φ and $R[s]\}$
$Z \subseteq \mu\mathtt{DML}(\mathsf{P}) \times S$		$\forall$	Z
$(\eta_x x.\varphi, s)$		$-$	$\{(\varphi[\eta x\, \varphi/x], s)\}$

Table 6: Evaluation game for disjunctive formulas (closure version)

Most of the moves of the evaluation game speak for themselves (given the interpretation of $\alpha \bullet \Phi$ as $(\bigwedge \alpha) \wedge \nabla\Phi$).

What makes the evaluation game for disjunctive formulas special is the kind of move that $\exists$ makes at a position of the form $(\nabla\Phi, s)$: here she picks a *relation* $Z \subseteq \mu\mathtt{DML}(\mathsf{P}) \times S$ of witnesses (with the requirement that Z is *full* on Φ and $R[s]$). Such a binary relation Z thus forms a new type of position, which is not a formula-state pair, but rather, a set of such pairs. These relational positions all belong to $\forall$, and his task at a position Z is simply to pick a witness from Z, that is, a pair (ψ, t) in Z. Of course this is in accordance with the semantic meaning of the cover modality.

In the following definition and propositions we isolate the key game-theoretic property of disjunctive formulas. Recall that, for a given strategy f in some evaluation game $\mathcal{E}(\xi, \mathbb{S})$ starting at position (ξ, s), we call a position (φ, t) f-reachable if there is some f-guided match in which the position (φ, t) is reached. We say that the state t is f-reachable if there is some formula φ such that the position (φ, t) is f-reachable.

Definition 2.67 Let ξ be a disjunctive formula, and let $(\mathbb{S}, s)$ be a pointed model.

A strategy f for $\exists$ in the evaluation game $\mathcal{E}(\xi, \mathbb{S})@(\xi, s)$ is called *thin* if at every f-reachable position of the form $(\nabla\Phi, s)$, f picks a relation $Z \subseteq \Phi \times R[s]$ such that for every $t \in R[s]$ there is exactly one $\varphi \in \Phi$ such that $(\varphi, t) \in Z$.

A strategy f for $\exists$ in $\mathcal{E}(\xi, \mathbb{S})@(\xi, s)$ is *thin* if for every $t \in S$ there is at most one formula φ of the form $\alpha \bullet \Phi$ such that (φ, t) is f-reachable.

If f is a thin strategy which is *winning* for $\exists$ in $\mathcal{E}(\xi, \mathbb{S})@(\xi, s)$ then we say that ξ is *strongly satisfied* in $\mathbb{S}$ at s, notation: $\mathbb{S}, s \Vdash_s \xi$. $\lhd$

The name 'separating' is chosen for obvious reasons: if, at position $(\nabla\Phi, s)$, $\exists$ picks a functional relation Z, she effectively separates the elements of Φ from one another, in the sense that there are no two witnesses $(\varphi, t), (\varphi', t)$ in Z with $\varphi \neq \varphi'$. It is easy to see that separating winning strategies on tree models are thin.

Proposition 2.68 *Let ξ be a disjunctive formula, and let $(\mathbb{S}, s)$ be a tree model. If f is a separating winning strategy for $\exists$ in $\mathcal{E}(\xi, \mathbb{S})@(\xi, s)$ then f is thin.*

Strong satisfaction is a very strong kind of satisfaction indeed, and in later chapters we will use it as a key model-theoretic tool. The thinness of separating strategies on tree models will turn out to be an extremely useful property. The fundamental model-theoretic property of disjunctive formulas is that without loss of generality we may always assume that winning strategies are separating, provided that we allow ourselves to move to a bisimilar model.

Theorem 2.69 *Let ξ be a disjunctive formula, and let $(\mathbb{S}, s)$ be a pointed model. Then the following are equivalent:*

1) $\mathbb{S}, s \Vdash \xi$

2) $\mathbb{S}', s' \Vdash_s \xi$ *for some pointed tree model such that* $\mathbb{S}, s \leftrightarroweq \mathbb{S}', s'$.

▶ `The proof of this theorem, which hinges on the semantics of the cover modality, will be given later.`

Since the cover modality can be expressed in terms of the box and diamond operators, it is obvious that $\mu\mathtt{DML}$ can be thought of as a *fragment* of the full language of the μ-calculus. One of the fundamental theorems in the theory of the modal μ-calculus is that $\mu\mathtt{DML}$ has the *same* expressive power as the full language. This equivalence is in fact effective, as stated by the next theorem.

Theorem 2.70 *There are effective procedures transforming an arbitrary formula $\varphi \in \mu\mathtt{ML}$ into an equivalent disjunctive formula, and vice versa. As a corollary, the languages $\mu\mathtt{ML}$, $\mu\mathtt{ML}_\nabla$ and $\mu\mathtt{DML}$ all have the same expressive power.*

The *proof* of Theorem 2.70 will be given in a later chapter.