Topics in Modal Logic: Exam

27 October 2017

- Each question is worth 20 points.
- Your grade will be based on your result on Exercise 7, together with the set of your best four results on the other exercises.
- Good luck!

Exercise 1 Let T be some set functor, and let $(L, \Vdash)$ be an invariant and expressive logic for T. Show that T admits a final coalgebra.

Exercise 2 Let $\mathbb{X} = (X, \xi)$, $\mathbb{S}_0 = (S_0, \sigma_0)$ and $\mathbb{S}_1 = (S_1, \sigma_1)$ be coalgebras for some set functor T, and let $f_0 : \mathbb{X} \to \mathbb{S}_0$ and $f_1 : \mathbb{X} \to \mathbb{S}_1$ be coalgebra morphisms.

(a) Show that the relation

$$\{(f_0 x, f_1 x) \in S_0 \times S_1 \mid x \in X\}$$

is a bisimulation between $\mathbb{S}_0$ and $\mathbb{S}_1$.

(b) Show that every bisimulation is of this form.

Exercise 3 Let C and D be nonempty finite sets (alphabets) such that $D \subseteq C$. Recall that $2 \times Id^C$ and $2 \times Id^D$ are the functors associated with deterministic C-automata and D-automata, respectively.

Given a finite word $u \in C^*$, define $u \restriction_D \in D^*$ to be the word obtained by deleting all letters from u that do not belong to D.

(a) Give an inductive definition of the operation $(\cdot) \restriction_D : C^* \to D^*$.

For an arbitrary C^*-language, we define $f_D(L) := \{u \restriction_D \mid u \in L\}$.

(b) Give a coinductive definition of the operation $f_D : P(C^*) \to P(D^*)$, and argue that your answer is correct.

Exercise 4 Let T be a standard and smooth set functor (that is, T preserves both inclusions and weak pullbacks). Recall that, given a set $\Phi \in TPS$, we define

$$\lambda^T_S(\Phi) := \{\tau \in TS \mid (\tau, \Phi) \in \overline{T}(\in_S)\},$$

where $\in_S \subseteq S \times PS$ is the membership relation on the set S. Show that λ^T constitutes a *distributive law*, i.e., it is a natural transformation $\lambda^T : T\breve{P} \dot{\to} \breve{P}T$.

Exercise 5 Let λ be a unary predicate lifting for a set functor T, and suppose that λ is completely multiplicative, i.e.

$$\lambda_S\big(\bigcap_{i\in I} X_i\big) = \bigcap_{i\in I} \lambda_S(X_i),$$

for all sets S and collections $\{X_i \mid i \in I\}$ of subsets of S. Show that the collection of maps

$$\lambda^\circ_S : \tau \mapsto \bigcap\{X \in PS \mid \tau \in \lambda_S(X)\}$$

defines a predicate lifting

$$\lambda^\circ : T \dot{\to} P.$$

Exercise 6 Let Λ be a set of unary predicate liftings for a set functor T. Show that the logic ML_Λ is invariant under behavioural equivalence.

Exercise 7 Let Λ be a set of predicate liftings for a set functor T, and let $\mathbf{H}$ be a one-step derivation system for Λ. If $\mathbf{H}$ is one-step sound and complete for T, then

$$\mathbf{H} \text{ is a complete derivation system for the set of } \mathrm{ML}_\Lambda\text{-validities on } \mathsf{Coalg}(T). \tag{1}$$

(a) Give a proof sketch of (1) in maximally 300 words.

(b) Make explicit where and how this proof uses the one-step completeness.