Topics in Modal Logic (Fall 2025)

Tutorial Exercises 1

Exercise 1 (Set functors)
Consider the following 'candidate functor' F which is a variation of the covariant power set functor. F maps a set S to its power set PS, and sends a function $f : S \to S'$ to the map $Ff : PS \to PS'$ given by

$$(Ff)(U) := S' \setminus (Pf)(S \setminus U).$$

Is this a functor on the category Set? Justify your answer by either providing a proof or a counterexample.

Exercise 2 (Distribution functor)
Show that the finitary distribution functor D_ω (Definition B6(f)) is indeed an endofunctor on the category Set. (The main points to check are (i) that $(D_\omega f)(\mu)$ is indeed a probability distribution on S' with finite support, for any $\mu \in D_\omega(S)$, and (ii) that $D(g \circ f) = (Dg) \circ (Df)$.)

Exercise 3 (Dynamical systems & isomorphisms)
A *dynamical system* is a coalgebra for the identity functor (on Set).

(a) Let $(S, \sigma), (S', \sigma')$ be two dynamical systems and let $f : (S, \sigma) \to (S', \sigma')$ be a bijective morphism. Show that f is an isomorphism (or *iso*, in categorical terms).

(b) For every $n \geq 1$, we let the dynamical system (C_n, γ_n) be defined by $C_n = \{0, \dots, n-1\}$ and $\gamma_n(k) = (k + 1) \mod n$.

Let (S, σ) be a dynamical system. An element $s \in S$ is said to have *period* n if $\sigma^n(s) = s$. For any $n \geq 1$, show that there is a one-one correspondence between the elements of S with period n and morphisms of the form $f : (C_n, \gamma_n) \to (S, \sigma)$.

(c) Does part (a) hold for T-coalgebras with respect to any set functor T? Justify your answer with a proof or a counterexample.

Exercise 4 (Final Kripke model?)
In this exercise we consider standard modal logic. Recall that Kripke models over the set Q of proposition letters can be identified with coalgebras for the functor $K_{P\mathsf{Q}} \times P$.

A set Φ of formulas is called *satisfiable* if there is a pointed Kripke model $\mathbb{S}$ in which Φ is satisfiable (in the sense that there is a state s such that $\mathbb{S}, s \Vdash \varphi$ for all $\varphi \in \Phi$). Such a set Φ is *maximally satisfiable* or an *MSS* if it is satisfiable itself, but it has no proper extensions that are satisfiable.

Suppose that we turn the collection $\mathcal{C}$ of MSSs into a Kripke model by putting

$$\begin{array}{lll} V(q) & := & \{\Gamma \in \mathcal{C} \mid q \in \Gamma\} \\ R & := & \{(\Gamma, \Delta) \mid \Diamond\Delta \subseteq \Gamma\}, \end{array}$$

where $\Diamond\Delta := \{\Diamond\delta \mid \delta \in \Delta\}$. Is this structure final in the category of Kripke models (seen as coalgebras)?

Exercise 5 (functional bisimulations) Let $\mathbb{S} = (S, \sigma)$ and $\mathbb{S}' = (S', \sigma')$ be two T-systems, and let $f : S \to S'$ be some map. Show that f is a coalgebra morphism from $\mathbb{S}$ to $\mathbb{S}'$ iff its graph $\mathsf{Gr}f := \{(s, s') \in S \times S' \mid s' = fs\}$ is a bisimulation between $\mathbb{S}$ and $\mathbb{S}'$.