Topics in Modal Logic (Fall 2025)

Tutorial Exercises 2

Exercise 1 (terminal sequence and final coalgebras)
An ω-*chain* in a category $\mathbb{C}$ is a diagram of the form

$$A_0 \xleftarrow{p_0^1} A_1 \xleftarrow{p_1^2} A_2 \xleftarrow{p_2^3} A_3 \longleftarrow \cdots \tag{1}$$

For the definition of a *limit* of such a diagram we refer to Appendix B of the lecture notes.

(a) Show that in the category Set every ω-chain has a limit. Hint: given the diagram (1), consider the set of all sequences $(a_n)_{n\in\omega}$ such that for each n we have $a_n \in A_n$ and $p_n^{n+1}(a_{n+1}) = a_n$.

Fix a set functor T. It follows from (a) that the ω-chain consisting of the initial part of the terminal sequence has a limit Z_ω:

$$\begin{tikzcd} Z_0 = 1 & Z_1 = T1 \arrow[l, "p_0^1 = !"'] & Z_2 = T^2 1 \arrow[l, "p_1^2 = T!"'] & \arrow[l] \cdots & Z_\omega \arrow[llll, bend right, "p_0^\omega"'] \arrow[lll, bend right, "p_1^\omega"'] \arrow[ll, bend right, "p_2^\omega"'] \end{tikzcd} \tag{2}$$

(b) For any T-coalgebra $\mathbb{S} = (S, \sigma)$ we may inductively define a ω-sequence of functions $\mathsf{beh}_n : S \to Z_n$ such that beh_0 is fixed by the finality of Z_0 in Set, and $\mathsf{beh}_{n+1} := (T\mathsf{beh}_n) \circ \sigma$:

$$\begin{tikzcd} \cdots & Z_n & Z_{n+1} \arrow[l, "p_n^{n+1}"'] & \cdots \\ & S \arrow[u, "\mathsf{beh}_n"] \arrow[r, "\sigma"'] \arrow[ur, dotted, "\mathsf{beh}_{n+1}"'] & TS \arrow[u, "T\mathsf{beh}_n"'] & \end{tikzcd} \tag{3}$$

Prove that $\mathbb{S}, s \simeq \mathbb{S}', s'$ implies $\mathsf{beh}_n(s) = \mathsf{beh}_n(s')$ for all n.

(c) Show that for any T-coalgebra $\mathbb{S} = (S, \sigma)$ there is a unique map $\mathsf{beh}_\omega : S \to Z_\omega$ such that $\mathsf{beh}_n = p_n^\omega \circ \mathsf{beh}_\omega$, for all n. Prove that $\mathbb{S}, s \simeq \mathbb{S}', s'$ implies $\mathsf{beh}_\omega(s) = \mathsf{beh}_\omega(s')$.

Let T be an endofunctor on some category $\mathbb{C}$. We say that T is *continuous* or *preserves limits of ω-chains* if for any diagram of the form (1) which has a limit $(A, (p_n)_{n\in\omega})$, the cone $(TA, (Tp_n)_{n\in\omega})$ is a limit of the diagram we obtain by applying the functor T to (1):

$$TA_0 \xleftarrow{Tp_0^1} TA_1 \xleftarrow{Tp_1^2} TA_2 \xleftarrow{Tp_2^3} TA_3 \longleftarrow \cdots \tag{4}$$

(d)* Let T be some continuous set functor. Show that T admits a final coalgebra. Hint: as the carrier of this coalgebra you may take the limit Z_ω of the ω-chain consisting of the initial part of the terminal sequence.

Exercise 2 (operations on streams)
Let T be the set functor $T := K_C \times Id$. Analogous to the example of the zip function (given by $\mathsf{zip}(\alpha, \beta) = (\alpha(0), \beta(0), \alpha(1), \beta(1), \cdots))$, we can define a binary operation on streams by considering the T-coalgebra $(C^\omega \times C^\omega, \sigma)$, with

$$\sigma(\alpha, \beta) := \big(\mathsf{h}(\alpha), (\mathsf{t}(\beta), \mathsf{t}(\alpha))\big).$$

Let $\mathsf{alt} : C^\omega \times C^\omega \to C^\omega$ be the (unique) coalgebra morphism from this coalgebra to the stream coalgebra $(C^\omega, \mathsf{h}, \mathsf{t})$. Which map is defined by alt? Show that your answer is correct.

Exercise 3 (zipping streams)
Prove the identity (15) in Example 2.19; that is, show that

$$\mathsf{zip}(\mathsf{e}(\alpha), \mathsf{q}(\alpha)) = \alpha,$$

for every stream $\alpha \in C^\omega$.

Exercise 4 (tight shuffle)
The *tight shuffle* of two languages $K, L \in P(C^*)$ is defined as

$$K \updownarrow L := \{a_1 b_1 \cdots a_n b_n \mid a_1 \cdots a_n \in K, b_1 \cdots b_n \in L\}. \tag{5}$$

(Here the a_i, b_j denotes letters, i.e., elements of C.) Give a coinductive definition[1] of this operation, and show that your definition is correct.
(Hint: similar to Example 2.17, you may want to introduce a set of expressions in a formal language which contains more connectives than just $\updownarrow$.)

Exercise 5 (induction via initiality)
For any endofunctor $T : \mathsf{C} \to \mathsf{C}$, a T-algebra is the categorical dual of a T-coalgebra. That is, a *T-algebra* consists of a pair (A, α) where A is an object and $\alpha : TA \to A$ is an arrow in C. A *T-algebra morphism* $f : (A, \alpha) \to (A', \alpha')$ is a C-arrow $f : A \to A'$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\alpha'} \\ TA & \xrightarrow{Tf} & TA' \end{array} \tag{6}$$

A T-algebra $\mathbb{A}$ is *initial* if for every T-algebra $\mathbb{B}$ there is a unique T-algebra morphism from $\mathbb{A}$ to $\mathbb{B}$.

In this exercise we consider the set functor $F := K_{\mathbb{1}} + Id$, where $\mathbb{1}$ is an arbitrary but fixed singleton set $\mathbb{1} = \{*\}$. Recall that F maps a set S to the disjoint union $\mathbb{1} \uplus S$.

(a) Show that the initial F-algebra is given as the structure $\mathbb{N} = (N, \nu)$, where N is the collection of natural numbers, and $\nu = [0, S] : \mathbb{1} + \omega \to \omega$ is the function that maps $*$ to 0 and an arbitrary natural number n to its successor $S(n) := n + 1$. (Recall from Definition B.14 that we use square brackets for the mediating arrows for coproducts.)

(b) Let $P \subseteq N$ be a subset of the natural numbers which contains 0 and which is closed under taking successors. Show how the inclusion map $\iota : P \hookrightarrow N$ can be made into a T-algebra morphism.

(c) Use initiality of $\mathbb{N}$ to show that $P = N$.

[1] In particular, you may not use the definition (5).