Topics in Modal Logic (Fall 2025)

Tutorial Exercises 3

Exercise 1 (limits and colimits in Set)

(a) Show that in the category Set every diagram has a limit. Hint: given the diagram $D : \mathsf{J} \to \mathsf{C}$, consider a suitable subset of the cartesian product $\prod_{j \in J} D(j)$.

(b) Show that in the category Set every diagram has a colimit. Hint: given the diagram $D : \mathsf{J} \to \mathsf{C}$, consider a suitable quotient of the disjoint union $\coprod_{j \in J} D(j)$.

Exercise 2 (bisimilarity implies behavioural equivalence) Prove that bisimilarity implies behavioural equivalence, for an arbitrary set functor T (that is, without assuming that T has a final coalgebra).

Exercise 3 (relation lifing) Prove the cases for product and coproduct of Proposition 3.9.

Exercise 4 (colimits in $\mathsf{Coalg}(T)$) Let T be a set functor. Show that the category $\mathsf{Coalg}(T)$ has all colimits, and that these colimits are in fact based on the underlying colimits in Set (cf. Remark 4.14).

Exercise 5 (class operations)

(a) Supply the details of the proof of Proposition 4.18 stating that $\mathsf{HS}(\mathsf{K}) \subseteq \mathsf{SH}(\mathsf{K})$, for an arbitrary class K of T-coalgebras.

(b) Prove that $\Sigma\mathsf{S}(\mathsf{K}) \subseteq \mathsf{S}\Sigma(\mathsf{K})$, for an arbitrary class K of T-coalgebras.

Exercise 6 (smooth functors and functoriality of $\overline{T}$) Let T be a smooth set functor. Prove that $\overline{T}R \,;\, \overline{T}Q \subseteq \overline{T}(R \,;\, Q)$.