Topics in Modal Logic (Fall 2025)

Tutorial Exercises 4

Exercise 1 (finite tree model property) Let $\mathbb{S} = (S, \sigma)$ be a T-coalgebra, for some smooth and standard set functor T. We say that a relation $R \subseteq S \times S$ *supports* σ if we have $\sigma(s) \in T(R(s))$, for all $s \in S$. We say that (S, σ) is a *tree coalgebra* if it is supported by a relation R for which (S, R) is a tree.[1] Similar definitions apply to coalgebra models; in particular we call the pointed coalgebraic model (S, σ, V, s) a (coalgebraic) *tree model* if its underlying coalgebra (S, σ) is a tree coalgebra with root s.

(a) Show that for every pointed coalgebra $(\mathbb{S}, s)$ there is a tree coalgebra $\mathbb{S}^*$ with root s^*, and a homomorphism $f : \mathbb{S}^* \to \mathbb{S}$ such that $f(s^*) = s$. Hint: consider the set S^*_s of all finite sequences u over S such that $\mathsf{first}(u) = s$.

(b) Show that ML_T has the *tree model property*. That is, show that any satisfiable formula $a \in \mathrm{ML}_T$ is satisfiable at the root of a tree model.

(c) Assuming that T is finitary, show that any satisfiable formula $a \in \mathrm{ML}_T$ is satisfiable at the root of a finitely branching tree model.

(d) Assuming that T is finitary, show that any satisfiable formula $a \in \mathrm{ML}_T$ is satisfiable at the root of a finite, finitely branching tree model.

Exercise 2 (witnessing truth) Let $(\mathbb{S}, s_0)$ be a pointed T-model , for some smooth and standard set functor T, and let $a_0 \in \mathrm{DML}_T$ be some formula. A relation $W \subseteq S \times \mathtt{Sfor}(a_0)$ is called a *witnessing relation* for the pair (s_0, a_0) if it satisfies

(i) $(s_0, a_0) \in W$;

(ii) $(s, a) \in W$ implies $\mathbb{S}, s \Vdash a$, if a is atomic;

(iii) $(s, a \vee b) \in W$ implies $(s, a) \in W$ or $(s, b) \in W$; and

(iv) $(s, (\bigwedge \pi) \bullet \nabla\alpha) \in W$ implies $\{s\} \times \pi \subseteq W$ and $(\sigma(s), \alpha) \in \overline{T}\big(W{\restriction_{S \times Base(\alpha)}}\big)$.

We write $\mathbb{S}, s_0 \Vdash_w a_0$ if there is a witnessing relation for (s_0, a_0).

Show that $\mathbb{S}, s_0 \Vdash a_0$ iff $\mathbb{S}, s_0 \Vdash_w a_0$.

Exercise 3 (second modal distributive law)

(a) Let Φ be a collection of sets of ML_∇-formulas. We let $\overline{P}{\in} \subseteq P\mathrm{ML}_\nabla \times PP\mathrm{ML}_\nabla$ denote the *lifted* membership relation, that is: $\alpha\ (\overline{P}{\in})\ \Phi$ if for all $a \in \alpha$ there is an $A \in \Phi$ such that $a \in A$, and for all $A \in \Phi$ there is a $a \in \alpha$ such that $a \in A$. In other words: $\alpha\ (\overline{P}{\in})\ \Phi$ means that α is a subset of $\bigcup \Phi$ which overlaps with every $A \in \Phi$. Furthermore, we define $(P\bigvee)(\Phi) := \{\bigvee \Phi \mid \Phi \in \Phi\}$.

Prove the *second* modal distributive law; that is, show that

$$\nabla(P\bigvee)(\Phi) \equiv \bigvee \Big\{\nabla\alpha \mid \alpha\ (\overline{P}{\in})\ \Phi\Big\} \tag{1}$$

[1] See Appendix A for the definition of a tree. Note that a tree (S, R) has a unique root.

(b) Show that the *propositional* distributive law:

$$\bigwedge(P\bigvee)(\Phi) \equiv \bigvee \left\{ \bigwedge\alpha \mid \alpha \; (\overline{P}\in) \; \Phi \right\} \tag{2}$$

can be seen as a special case of (1).

(c) What can you conclude from the validity of (1) about normal forms for standard modal logic?

Exercise 4 (relation lifting for the bag functor) In this exercise we consider relation lifting for the finitary *bag functor* B_ω. *For the sake of lightweight notation we write* B *instead of* B_ω. Recall that

$$BS := \{\mu : S \to \mathbb{N} \mid |\mathsf{supp}(\mu)| < \omega\},$$

where we define the *support* of a function $\mu : S \to \mathbb{N}$ as $\mathsf{supp}(\mu) := \{s \in S \mid \mu(s) > 0\}$. Elements of $B(S)$ will be called finite *bags*.

We can describe the action of the functor B on a map $f : S \to S'$ as follows:

$$(Bf)(\mu)(s') := \sum_{s \in f^{-1}(s')} \mu(s).$$

Relation lifting for this functor can be described in the same way as for the distribution functor D (cf. Example 5.13 of the lecture notes).

(a) Given a relation $R \subseteq S_0 \times S_1$, show that $\overline{B}(R)$ consists of those pairs $(\mu_0, \mu_1) \in BS_0 \times BS_1$ for which there is a finite bag $\rho : R \to \mathbb{N}$ in $B(R)$ which satisfies the 'magic square condition':

$$\begin{aligned} \text{for all } s_0 \in S_0. \quad & \mu_0(s_0) = \sum\{\rho(s_0, y_1) \mid (s_0, y_1) \in R\}, \\ \text{and for all } s_1 \in S_1. \quad & \mu_1(s_1) = \sum\{\rho(y_0, s_1) \mid (y_0, s_1) \in R\}. \end{aligned}$$

(Hint: this is not difficult; the point is to clarify the role of the maps $B\pi_i : BR \to BS_i$, where the $\pi_i : R \to S_i$ are the projection maps.)

(b) Prove that B preserves weak pullbacks.

(c) Show that for any relation $R \subseteq S_0 \times S_1$ and any pair of finite bags $\mu_i \in B(S_i)$, we have that

$$(\mu_0, \mu_1) \in \overline{B}(R) \text{ implies } |\mu_0| = |\mu_1|,$$

where the *weight* $|\alpha|$ of a weight function $\alpha \in B(A)$ is defined as $|\alpha| := \sum_{a \in A} \alpha(a)$.)

Exercise 5 (*)**(distributive laws and the lifted membership relation)** Let $T : \mathsf{Set} \to \mathsf{Set}$ be smooth. Show that $\lambda^T = \{\lambda^T_X\}_{X \in \mathsf{Set}}$ is a distributive law of T over *both* the covariant and the contravariant power set functor P (cf. Remark 5.25).