Contents

1 Stone's Representation Theorem

1.1 Stone's Representation Theorem

Stone's Representation Theorem

Theorem 1.1. *The category of Boolean algebras (BA) is dual to the category of Stone spaces (SSpa).*

1.2 Boolean Algebras

Boolean Algebras

Definition 1.2. A sextuple $B = (B, \wedge, \vee, \neg, \top, \bot)$ with

- B is a set
- $\top, \bot \in B$
- $\wedge : B \times B \longrightarrow B$ the meet (or greatest lowerbound)
- $\vee : B \times B \longrightarrow B$ the join (or smallest upperbound)
- $\neg : B \longrightarrow B$ the negation

such that for all $a, b, c \in B$ we have

Lattice

- $a = a \wedge a = a \vee a = a \wedge \top = a \vee \bot$
- $a \wedge b = b \wedge a$
- $a \vee b = b \vee a$

- $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- $a \vee (b \vee c) = (a \vee b) \vee c$
- $a \wedge (a \vee b) = a$
- $a \vee (a \wedge b) = a$

Distrubitive

- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Negation

- $a \wedge \neg a = \bot$
- $a \vee \neg a = \top$

Morphisms Between BA

Definition 1.3. A function $f : A \longrightarrow B$ between BA is a morphism if

- $f(a \wedge b) = f(a) \wedge f(b)$
- $f(a \vee b) = f(a) \vee f(b)$
- $f(\neg a) = \neg f(a)$
- $f(\top) = \top$
- $f(\bot) = \bot$

Remark 1.1. If f is a BA-morphism, then f is also order preserving. But if f is order preserving, then f is not neccessarily a BA-morphism.

1.3 Stone Spaces

Stone Spaces

Definition 1.4. A Stone space is a topological space $(X, \mathcal{T})$ such that $(X, \mathcal{T})$ is compact, Hausdorff and zero-dimensional.

Remark 1.2. Equivalent definitions are:

- compact, Hausdorff and totally disconnected
- compact, Hausdorff and a basis of clopens
- many others...

Zero-dimensional

Definition 1.5. Let $(X,\mathcal{T})$ be a topological space and let $X=\bigcup U_i$ be an open cover. Then a refinement of $X=\bigcup U_i$ is an open cover $X=\bigcup V_j$, such that for every j there exists an i such that $V_j\subseteq U_i$.

Definition 1.6. A topological space $(X,\mathcal{T})$ is n-dimensional if for every open cover $X=\bigcup U_i$ has a refinement $X=\bigcup V_j$ such that every $x\in X$ is contained in at most $n+1$ sets from $\{V_j\}_j$.

Lemma 1.7. 1. *T_1 + zero-dimensional $\Rightarrow$ the clopens form a basis*

2. *compact + a basis of clopens $\Rightarrow$ zero-dimensional*

1.4 The Functor $G:\mathsf{SSpa}\longrightarrow\mathsf{BA}$

The Functor $G:\mathsf{SSpa}\longrightarrow\mathsf{BA}$

Definition 1.8. Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be Stone spaces. Let $f:X\longrightarrow Y$ be continuous. Then we define $G:\mathsf{SSpa}\longrightarrow\mathsf{BA}$ by

- $G(X,\mathcal{T})=(\mathcal{B}_{\mathcal{T}},\cap,\cup,X\setminus\bullet,X,\emptyset)$ with $\mathcal{B}_T:=\{U\in\mathcal{T}\mid X\setminus U\in\mathcal{T}\}$
- $G(Y,\mathcal{U})=(\mathcal{B}_U,\cap,\cup,X\setminus\bullet,X,\emptyset)$ with $\mathcal{B}_U:=\{V\in\mathcal{U}\mid X\setminus V\in\mathcal{U}\}$
- $G(f)=f^{-1}|_{\mathcal{B}_{\mathcal{U}}}$

1.5 The Functor $F:\mathsf{BA}\longrightarrow\mathsf{SSpa}$

Filters And Ideals

Definition 1.9. A subset $F\subseteq B$ is called a filter over B if

- $\top\in F$
- $a,b\in F\Rightarrow a\wedge b\in F$
- $a=a\wedge b$ and $a\in F\Rightarrow b\in F$

F is called an ultrafilter if F is a filter and

- $\bot\notin F$
- $a\in F$ or $\neg a\in F$

Definition 1.10. A subset $I\subseteq B$ is called an ideal over B if

- $\bot\in I$
- $a,b\in I\Rightarrow a\vee b\in I$
- $a=a\vee b$ and $a\in I\Rightarrow b\in I$

I is called a prime ideal if I is an ideal and

- $\top \notin I$
- $a \wedge b \in I \Rightarrow a \in I \text{ or } b \in I$

Lemma 1.11. *Every subset $A \subseteq B$ generates a filter by taking the intersection of all filters F such that $A \subseteq F$.*

Lemma 1.12. *If $\{F_i\}_I$ are filters of B such that $F_i \subseteq F_{i+1}$ for all $i \in I$, then $F = \bigcup_I F_i$ is also a filter of B*

Lemma 1.13. *If $\{a_i\}_I$ are elements of B such that for all finite subsets $J \subseteq I$ we have $\bigwedge_J a_i \neq \bot$ (the finite meet property), then $\{a_i\}_I$ generates a proper filter*

Lemma 1.14. *Every proper filter F can be extended to an ultrafilter.*

An Isomorphism

$$\mathcal{U}\mathsf{f}(B) := \{F \subset B \mid F \text{ is an ultrafilter }\}$$

We define $\rho : B \longrightarrow \mathcal{P}(\mathcal{U}\mathsf{f}(B))$ by

$$\rho(a) = \{F \in \mathcal{U}\mathsf{f}(B) \mid a \in F\}$$

Lemma 1.15. *The set $\{\rho(a) \mid a \in B\}$ is a basis of clopens*

The Stone Space

Let $\mathcal{T}$ be the topology generated by $\{\rho(a) \mid a \in B\}$.

Theorem 1.16. *$(\mathcal{U}\mathsf{f}(B), \mathcal{T})$ is a Stone space*

- compact $\surd$
- Hausdorff $\surd$
- a basis of clopens $\surd$

The Functor $F : \mathsf{BA} \longrightarrow \mathsf{SSpa}$

Let A, B be BA. Let $\mathcal{T}$ be the topology generated by $\{\rho(a) \mid a \in A\}$ and let $\mathcal{U}$ be the topology generated by $\{\rho(b) \mid b \in B\}$. Let $\varphi : A \longrightarrow B$ be a BA morphism. Then we define $F : \mathsf{BA} \longrightarrow \mathsf{SSpa}$ by

- $F(A) = (\mathcal{U}\mathsf{f}(A), \mathcal{T})$
- $F(B) = (\mathcal{U}\mathsf{f}(B), \mathcal{U})$
- $F(\varphi)(S) = \varphi^{-1}(S)$ for $S \in \mathcal{U}\mathsf{f}(B)$

Remark 1.3. For $a \in A$ we have that $F(\varphi)^{-1}(\rho(a)) = \rho(\varphi(a))$.

1.6 Stone's Representation Theorem

$GF \cong \mathsf{id}_{\mathsf{BA}}$

Let A, B be BA and let $\varphi : A \longrightarrow B$ be a BA morphism. Then

$$GF(\varphi : A \longrightarrow B) = \varphi^* : \{\rho(a) \mid a \in A\} \longrightarrow \{\rho(b) \mid b \in B\}$$

with $\varphi^*(\rho(a)) = \rho(\varphi(a))$

An Isomorphism

Let $(X, \mathcal{T})$ be a Stone space. Let $\mathcal{B}_\mathcal{T}$ denote the basis of all clopens of $\mathcal{T}$. Then we can define $\pi : X \longrightarrow \mathcal{U}\mathsf{f}(\mathcal{B}_\mathcal{T})$ by

$$\pi_x := \pi(x) := \{U \in \mathcal{B}_T \mid x \in X\}$$

We can extend π to the topology, i.e. $\pi : \mathcal{B}_T \longrightarrow \{\rho(U) \mid U \in \mathcal{B}_T\}$ by

$$\pi(U) := \{\pi_x \in \mathcal{U}\mathsf{f}(\mathcal{B}_T) \mid x \in U\}$$

Lemma 1.17. *$\pi : X \longrightarrow \mathcal{U}\mathsf{f}(\mathcal{B}_\mathcal{T})$ and $\pi : \mathcal{B}_T \longrightarrow \{\rho(U) \mid U \in \mathcal{B}_T\}$ are isomorphisms.*

- $x \neq y \Rightarrow \pi_x \neq \pi_y$ $\surd$
- if $S \in \mathcal{U}\mathsf{f}(\mathcal{B}_\mathcal{T})$, then $S = \pi_x$ for some $x \in X$ $\surd$
- $\{\pi(U) \mid U \in \mathcal{B}_T\} = \{\rho(U) \mid U \in \mathcal{B}_T\}$ $\surd$
- π is continuous and open $\surd$

$FG \cong \mathsf{id}_{\mathsf{SSpa}}$

Let $(X, \mathcal{T}), (Y, \mathcal{U})$ be Stone spaces and let $f : X \longrightarrow Y$ be continuous. Then

$$FG(f : X \longrightarrow Y) = f^* : \{\pi_x \mid x \in X\} \longrightarrow \{\pi_y \mid y \in Y\}$$

with $f^*(\pi_x) = \pi_{f(x)}$ and ${f^*}^{-1} : \{\pi(V) \mid V \in \mathcal{B}_\mathcal{U}\} \longrightarrow \{\pi(U) \mid U \in \mathcal{B}_\mathcal{T}\}$ is

$${f^*}^{-1}(\pi(V)) = \pi(f^{-1}(V))$$

2 An Extension To Modal Algebras

2.1 The Theorem

Theorem 2.1. *The category of Modal Algebras (MAl) is dual to the category of Descriptive Frames (DFra)*

2.2 Modal Algebras

Definition 2.2. Let A, B be Boolean algebras. Then $\tau : A \longrightarrow B$ is a hemimorphism if

- $\tau(\top) = \top$
- $\tau(a \wedge b) = \tau(a) \wedge \tau(b)$

Definition 2.3. A modal algebra is a pair (A, τ) where A is a Boolean algebra and $\tau : A \longrightarrow A$ is a hemimorphism from A into A

Definition 2.4. Let (A, τ) and (B, σ) be modal algebras, then a BA-morphism $\varphi : A \longrightarrow B$ is a MAl-morphism if

$$\sigma \circ \varphi = \varphi \circ \tau$$

2.3 Descriptive Frames

Definition 2.5. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Let $r \subseteq X \times Y$ be any relation. Define $r^* : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ by

$$r^* D := \{x \in X \mid \forall y \in Y : xry \Rightarrow y \in D\}$$

The relation r is called continuous if for all $V \in \mathcal{U}$ we have $r^* V \in \mathcal{T}$

Definition 2.6. A descriptive frame is a triple $(X, r, \mathcal{T})$ where $(X, \mathcal{T})$ is a Stone space and $r \subseteq X \times X$ is a continuous relation.

Definition 2.7. Let $(X, r, \mathcal{T})$ and $(Y, s, \mathcal{U})$ be descriptive frames. Then a continuous function $f : X \longrightarrow Y$ is a DFra-morphism if for all $x, y \in X$ we have

$$xry \Leftrightarrow f(x) s f(y)$$

2.4 The Functors

Let $(X, r, \mathcal{T})$ be a descriptive frame. We will extend the functor G from Stone's representation theorem to descriptive frames by

$$G(r) := r^*$$

where

$$r^* D := \{x \in X \mid \forall y \in X : xry \Rightarrow y \in D\}$$

Lemma 2.8. *Let $(X, r, \mathcal{T})$ be a descriptive frame. Let $\mathcal{B}_\mathcal{T}$ denote the set of all clopens of $\mathcal{T}$. Then $r^* : \mathcal{B}_\mathcal{T} \longrightarrow \mathcal{B}_\mathcal{T}$ is a hemimorphism*

Let (A, τ) be a modal algebra. We will extend the functor F from Stone's representation theorem to modal algebras by

$$F(\tau) := \tau_*$$

where

$$\tau_* \subseteq \mathcal{U}\mathsf{f}(A) \times \mathcal{U}\mathsf{f}(A) \text{ with } S \tau_* T \Leftrightarrow \forall a \in A : \tau a \in S \Rightarrow a \in T$$

Lemma 2.9. *$\tau_* \subseteq \mathcal{U}\mathsf{f}(A) \times \mathcal{U}\mathsf{f}(A)$ is a continuous relation with respect to $\{\rho(a) \mid a \in A\}$.*

Claim: for all $a \in A$ we have $(\tau_*)^* \rho(a) = \rho(\tau a)$

Let (A, τ) be a modal algebra. Then

$$GF(A, \tau) = (\{\rho(a) \mid a \in A\}, (\tau_*)^*)$$

with $(\tau_*)^* \rho(a) = \rho(\tau a)$.

Let $(X, r, \mathcal{T})$ be a descriptive frame. Then

$$FG(X, r, \mathcal{T}) = (\{\pi_x \mid x \in X\}, (r^*)_*, \{\pi(U) \mid U \in \mathcal{B}_\mathcal{T}\})$$

with $\pi_x (r^*)_* \pi_y$ if and only if for all $U \in \mathcal{B}_\mathcal{T}$ we have $r^* U \in \pi_x$ implies $U \in \pi_y$.

Lemma 2.10. $\pi_x (r^*)_* \pi_y \Leftrightarrow xry$