Comparisons of Different Constructions in Algebraic K-Theory

Master Thesis

MAO Zhouhang

Advisors: Prof. Matthew MORROW

Department of Mathematics, Université Paris Diderot

Abstract

This thesis is devoted to summarize the algebraic K-theory in the modern ∞-categorical language. The algebraic K-theory, in short, is the higher dimensional generalization of the Grothendieck group K_0. In this thesis, we discuss two mainlines of the algebraic K-theory, and their comparison. The author claims no originality. A general reference for classic algebraic K-theory is [16]. More modern developments are found in [2], [7] and [8]. Further references are mentioned in the text.

Contents

Chapter 1

Introduction to K-Theory

The K_0-group originates from classifications of finite dimensional vector bundles over a topological space (such as an n-sphere), or classifications of finitely generated projective modules over a ring. Let's consider the case of finitely generated projective modules. Let R be a ring. There are two constructions of the Grothendieck group $K_0(R)$:

1. Isomorphism classes of finitely generated projective R-modules, along with the direct sum operation, form an abelian monoid. Take the group completion of this monoid, we obtain a group $K_0(R)$.
2. Another construction follows from the concept of short exact sequences. We consider the free abelian group generated by isomorphism classes of finitely generated projective R-modules, modulo the relation given by short exact sequences: for each short exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated projective R-modules, we impose a relation $[M] = [M'] + [M'']$.

Since each short exact sequence splits, we deduce that two constructions give the same K_0-group. However, these constructions rely on different structures on the category of finitely generated projective R-modules, namely the structure of the direct sum operation, and the structure of exact sequences. In this thesis, we will expose their generalizations to the K-theory space.

The structure of the direct sum operation generalizes to the concept of symmetric monoidal ∞-categories, which occupies Chapter 4, and the structure of exact sequences generalizes to the concept of ∞-categories with cofibrations, which occupies Chapter 2, or exact ∞-categories, which occupies Chapter 3.

To motivate the concept of the K-theory space, we only note that we have blindly omit the information about the deformation by taking isomorphism classes. That would be a great loss even if eventually we only want to study

K_0-groups. A parallel phenomenon leads to the great success of homological algebra, which was motivated by extending one-sided exact sequences. However, the K-theory space seems much more difficult than homological algebra, partially due to the homotopic nature of this topic. Nevertheless, we will sketch important constructions in the thesis, and furnish enough references. Two approaches, as in the case of K_0-groups, will lead to the same space under some mild conditions, which is the goal of Chapter 5. More classical constructions are summarized in Chapter 6.

Let's take a look at the K-theory space briefly. In short, as a result of all these K-theory space constructions, we will obtain an $\mathbb{E}_\infty$-group (equivalently, a spectral object in $\mathcal{K}$an, the ∞-category of Kan complexes, although we won't prove that the result is a spectrum for the K-theory originated from the structure of exact sequences). For an ∞-category with the structure of direct sum operation, the description seems clearer: we just impose that the "multiplication" operation of the $\mathbb{E}_\infty$-group mimics the direct sum operation. It's conceptually easier to understand that this is essentially just a group completion: to make the direct sum operation invertible. For an ∞-category with the structure of exact sequences (this will eventually be an ∞-category with cofibrations in Chapter 2), this seems a bit tougher to explain. Roughly speaking, we want to impose the "multiplication" of M' and M'' to be M whenever there is a short exact sequence $M' \to M \to M''$, just like the case of K_0. This will lead to the $S_\bullet$-construction. Our main theorem of comparison, Theorem 5.2, will claim that these two constructions will be homotopically coincide under a very mild condition. We need to make precise the concept of exact sequences, which is the goal of Chapter 2.

Let's sketch the historical development of higher K-theory. It was Quillen that first successfully defined (see [12, 13]) the higher K-groups of finitely generated projective R-modules from the $+$-construction: Later, he defined his famous Q-construction in [11]. The group completion construction appeared in [15], and Waldhausen came up with his $S_\bullet$-construction in [17]. Finally, Barwick rebuilt $S_\bullet$-construction and Q-construction in the language of ∞-categories, see [2, 1]. The relationship between these structures are summarized below:

- Exact K-theory

  1. For ∞-categories with cofibrations $\mathcal{C}$: Barwick's $S_\bullet$-construction $\Omega|S_\bullet\mathcal{C}|$ (Chapter 2).

  2. Special case of the case 1 for Waldhausen categories $\mathcal{C}$: Waldhausen's $S_\bullet$-construction $\Omega|S_\bullet\mathcal{C}|$;

  3. For exact ∞-categories $\mathcal{C}$: Barwick's Q-construction (Chapter 3) $\Omega Q\mathcal{C}$, the result of which coincide with Barwick's $S_\bullet$-construction;

4. Special case of the case 3 for exact categories $\mathcal{C}$: Quillen's Q-construction $\Omega Q\mathcal{C}$.

- Additive K-theory

  1. For symmetric monoidal categories: group completion construction (Chapter 4);

  2. Special case of the case 1 for a ring R: Quillen's $+$-construction $K_0(R) \times BGL(R)^+$.

For modules over a ring, all these constructions coincide.

The author claims no originality for any propositions or arguments. The K-Theory in modern language is sketched in Lurie's lecture notes [7] and scrupulously explained in Barwick's work [2, 1]. Classical works record the names of Quillen, Waldhausen and Segal, and a lot of other people. The topic is suggested and the thesis is supervised by Professor Matthew Morrow.

Chapter 2

∞-categories with Cofibrations

As seen in the introduction, Chapter 1, we want to precipitate the concept of an ∞-category with the structure of exact sequences. Roughly speaking, given a short exact sequence (in any sense) $M' \to M \to M''$, we should view M'' as a quotient M/M'. It would be reasonable for us to only record these morphisms $M' \to M$, which will be called cofibrations in the thesis. To recover the corresponding exact sequence from a cofibration $M' \to M$, we take the pushout of the cofibration $M' \to M$ along a collapsing morphism $M' \to *$, obtaining a triangle (see Definition B.26) served as an exact sequence which is usually called a cofiber sequence in the literature being distinguished from the dual concept of fiber sequences, both of which are considered "exact" in classical commutative algebra (because in classical commutative algebra, most underlying categories are stable or abelian, cofiber sequences and fiber sequences usually coincide), see Remark 3.10 for a bit further discussions. A reference for this chapter is [7, Lecture 18].

Definition 2.1 ([7, Lecture 18, Definition 5],[2, Definition 2.7]) *An ∞-category with cofibrations is a pointed ∞-category $\mathcal{C}$ with a collection of morphisms, called cofibrations, such that*

1. *All equivalences are cofibrations and the collection of cofibrations is closed under composition (which means that, for any 2-simplex $t \in \mathcal{C}_2$, if two faces $d_0t, d_2t \in \mathcal{C}_1$ are cofibrations, then so is the "composition" $d_1t \in \mathcal{C}_1$);*
2. *All maps $* \to X$ are cofibrations;*
3. *For a cofibration $f\colon X \to X'$ and an arbitrary morphism $X \to Y$, there exists a pushout square*

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y' \end{array}$$

and that g is a cofibration (therefore, by previous axioms, every co-base change of a cofibration exists and is a cofibration).

Example 2.2 *Given a (unital) ring R, Denote by $\mathcal{M}(R)$ the nerve of the additive category of finitely generated R-modules. $\mathcal{M}(R)$ is an ∞-category with cofibrations, where the cofibrations are given by injections.*

Proof Axiom 1 (note that the composition is unique here) and axiom 2 in Definition 2.1 are satisfied by construction. To show the axiom 3, we first note that arbitrary pushouts exist in $\mathcal{M}(R)$. Furthermore, given a pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X'' \arrow[d] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

of R-modules, if f is injective, so is g. This is due to the fact that pushouts of monomorphisms are also monomorphisms in an abelian category. □

By a similar argument we can show that

Example 2.3 *Denote by $\mathcal{P}(R) \subseteq \mathcal{M}(R)$ the full subcategory spanned by finitely generated projective R-modules. Then $\mathcal{P}(R)$ is an ∞-category with cofibrations, where the cofibrations are given by injections with a projective cokernel.*

Remark 2.4 *Examples 2.2 and 2.3 are special cases of exact categories. We will define exact ∞-categories in Chapter 3. On the other hand, note that all cofibrations in the example 2.3 are in fact split injections, which stimulates Proposition 2.5 and Example 2.6.*

Proposition 2.5 *In an ∞-category with cofibrations $\mathcal{C}$, every natural map of the form $g\colon Y \to X \vee Y$ is a cofibration.*

Proof

Step 1: There exists a pushout square

$$\begin{tikzcd} * \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "g"] & X \vee Y \end{tikzcd}$$

in $\mathcal{C}$;

Step 2: The top morphism $f\colon * \to X$ is a cofibration (Axiom 2 of Definition 2.1);

Step 3: Therefore g is a cofibration (Axiom 3 of Definition 2.1). □

Example 2.6 *Every pointed ∞-category which admits arbitrary finite coproducts could be made into an ∞-category with cofibrations by taking all maps equivalent to a split map $Y \to X \vee Y$ as cofibrations. This is the "minimal" collection of cofibrations by Proposition 2.5.*

Proof We check axioms in Definition 2.1:

Axiom 1: It follows from the homotopy-coherent associativity of coproducts: $(X \vee Y) \vee Z \simeq X \vee (Y \vee Z)$ in $\mathcal{C}$;

Axiom 2: Note that $* \to X \vee * \simeq X$;

Axiom 3: Given a pushout diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\ X \vee Y \arrow[r] & W \end{tikzcd}$$

We need to show that g is equivalent to a split map. We consider a larger diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\ X \arrow[r] & X \vee Y \arrow[r] & X \vee Z \end{tikzcd}$$

Both the square on the left and the outer square are pushout squares, therefore the square on the right is also a pushout square, which implies that g is equivalent to the split map $Z \to X \vee Z$. $\square$

Example 2.7 *As a counterpart of Example 2.6, every finitely cocomplete pointed ∞-category could be made into an ∞-category with cofibrations where all morphisms are cofibrations. This is the "maximal" collection of cofibrations.*

Definition 2.8 *Suppose $\mathcal{C}$ is an ∞-category with cofibrations. We define $K_0(\mathcal{C})$ as the abelian group defined by*

- *Generators: objects $X \in \mathcal{C}$;*
- *Relations: $[X'] + [X''] = [X]$ whenever there is a cofibration $X' \to X$ with a cofiber sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$, which means that there is a pushout diagram*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d, "g"] \\ * \arrow[r] & X'' \end{tikzcd}$$

Example 2.9 *Take $\mathcal{C}$ in examples 2.2 and 2.3, we recover the ordinary concept of Grothendieck groups.*

Remark 2.10 *We note that the group $K_0(\mathcal{C})$ is "automatically" abelian. To be more precise, the group determined by the same collection of generators and relations is abelian, therefore coincides with $K_0(\mathcal{C})$, by virtue of the cofiber sequences $X' \to X' \vee X'' \to X''$ and $X'' \to X' \vee X'' \to X'$.*

By the canonical cofiber sequence $* \to * \to *$, we have

Proposition 2.11 $[*] = 0$ *in* $K_0(\mathcal{C})$.

Remark 2.12 *In order to understand higher K-theory, let's consider a pointed Kan complex $(K, *)$. The loop space $L = \Omega K$ admits a "multiplication" defined by concatenating two paths. This "multiplication" is neither naturally defined (which depends on a choice of parametrizing), nor invertible, nor associative. At a first glance, one can recover the well-definedness, invertibility and associativity by taking homotopy classes of loops, namely $\pi_0(L) = \pi_1(K)$. However, this leads to a great loss of information: fundamental groups are far from characterizing the space. Later developments of homotopy theory lead us to consider homotopy coherency. This "multiplication" is in fact homotopy coherently associative and invertible. Basically speaking, we need to record homotopies underlying $abc \simeq (ab)c \simeq a(bc)$, these homotopies are compatible for different parenthesizing of $abcd \simeq (a(b(cd))) \simeq (a((bc)d)) \simeq \dots$.*

Our goal is to define a K-theory space $K(\mathcal{C}) = \Omega L$ for an ∞-category with cofibrations $\mathcal{C}$, such that $\pi_0 K(\mathcal{C}) = \pi_1(L) = K_0(\mathcal{C})$. Instead of imposing $[X'] + [X/X'] = [X]$, we furnish a 2-cell, imposing $[X'] + [X/X'] \simeq [X]$, where each object $X \in \mathcal{C}$ determines a loop $[X] \in \Omega L$. For a filtered object $X_1 \to X_2 \to X_3$, where $X_1 \to X_2$ and $X_2 \to X_3$ are cofibrations, we have four cofiber sequences: $X_1 \to X_2 \to X_2/X_1$, $X_1 \to X_3 \to X_3/X_1$, $X_2 \to X_3 \to X_3/X_2$ and $X_2/X_1 \to X_3/X_1 \to X_3/X_2$ (one can recall the octahedral axiom for triangulated categories). We furnish a 3-cell to combine these informations.

In general, in order to avoid technical difficulties, we follow Barwick's approach [2]:

Definition 2.13 *A filtered object of length m is a functor $\Delta^m \to \mathcal{C}$ such that the image of each edge is a cofibration. The ∞-category $\mathscr{F}_m(\mathcal{C})$ is defined as the full subcategory of* $\mathrm{Fun}(\Delta^m, \mathcal{C})$ *spanned by filtered objects of length m.*

Remark 2.14 *We discuss briefly about straightening and unstraightening process. A formal treatment and precise definitions could be found in [9, Chapter 3, Section 3.2]. Given an ∞-category $\mathcal{S}$, there is a correspondance between two worlds: on one hand, one has the collection of coCartesian fibrations over $\mathcal{S}$; on the other hand, one has the collection of classifying functors $\mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$. Given a coCartesian fibration $F\colon \mathcal{C} \to \mathcal{S}$, one can obtain a classifying functor $G\colon \mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$, such that for each $s \in \mathcal{S}$, $G(s)$ is categorically equivalent to $F^{-1}(s)$. This is called the straightening process. Given a classfying functor $G\colon \mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$, one can associate a coCartesian* *fibration $F\colon \mathcal{C} \to \mathcal{S}$ such that $F^{-1}(s)$ is categorically equivalent to $G(s)$. This is called the unstraightening process.*

The definition of coCartesian fibrations is quite technical:

Definition 2.15 *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. A morphism $f\colon c_1 \to c_2$ in $\mathcal{C}$ is p-coCartesian, or a p-coCartesian lift of $p(f)$ if the following map is an acyclic Kan fibration:*

$$\mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{D}_{p(c_1)/}} \mathcal{D}_{p(f)/}$$

Definition 2.16 *A functor $p\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a coCartesian fibration if the following two properties are satisfied:*

- *The functor p is an inner fibration (see Definition B.1);*
- *For every object $c_1 \in \mathcal{C}$ and every morphism $\alpha\colon p(c_1) = d_1 \to d_2$ in $\mathcal{D}$, there is a p-coCartesian lift $f\colon c_1 \to c_2$ of α.*

By unstraightening, we can obtain an ∞-category $\mathscr{F}(\mathcal{C})$ and a cocartesian fibration $\mathscr{F}(\mathcal{C}) \to \mathrm{N}(\Delta^{\mathrm{op}})$ classified by the functor $[m] \mapsto \mathscr{F}_m(\mathcal{C})$. The choice of $\mathscr{F}(\mathcal{C})$ is essentially unique. We follow Barwick's [2, Construction 5.3]:

Denote by M the ordinary category of pairs $([m] \in \Delta^{\mathrm{op}}, i \in [m])$ where a morphism $([n], j) \to ([m], i)$ is given by a morphism $\varphi\colon [m] \to [n]$ in Δ such that $j \leq \varphi(i)$. We have a natural functor $M \to \Delta^{\mathrm{op}}$ given by $([m], i) \mapsto [m]$.

We can construct the coCartesian fibration $\mathscr{F}(\mathcal{C}) \to \mathrm{N}(\Delta^{\mathrm{op}})$ by the universal property:

Proposition 2.17 *For any simplicial set K with a simplicial map $K \to \mathrm{N}(\Delta^{\mathrm{op}})$, we have a natural bijection between $\mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta/\mathrm{N}(\Delta^{\mathrm{op}})}}(K, \mathscr{F}(\mathcal{C}))$ and the set of simplicial maps $K \times_{\mathrm{N}(\Delta^{\mathrm{op}})} \mathrm{N}(M) \to \mathcal{C}$ such that for each edge (e, f) in $K \times_{\mathrm{N}(\Delta^{\mathrm{op}})} \mathrm{N}(M)$, if the image of f in $\mathrm{N}(\Delta^{\mathrm{op}})$ is an equivalence, then the image of (e, f) is a cofibration.*

We see that the fiber of $\mathscr{F}(\mathcal{C}) \to \mathrm{N}(\Delta^{\mathrm{op}})$ over $[m]$ is $\mathscr{F}_m(\mathcal{C})$ (not just up to a weak equivalence) by taking $K \to \mathrm{N}(\Delta^{\mathrm{op}})$ to be the constant functor at $[m]$. Hence we can understand an object in $\mathscr{F}(\mathcal{C})$ as a pair $([m] \in \Delta^{\mathrm{op}}, X \in \mathscr{F}_m)$.

Definition 2.18 *A filtered object $X\colon \Delta^m \to \mathcal{C}$ is called totally filtered if $X(0) = *$. Denote by $\mathscr{S}_m(\mathcal{C}) \subseteq \mathscr{F}_m(\mathcal{C})$ the full subcategory spanned by totally filtered objects. $\mathscr{S}(\mathcal{C}) \subseteq \mathscr{F}(\mathcal{C})$ is the full subcategory spanned by pairs $([m], X)$ where X is totally filtered.*

Come back to Waldhausen's $S_\bullet$-construction. Now we have a functor $\mathscr{S}(\mathcal{C}) \to \mathrm{N}(\Delta^{\mathrm{op}})$. By straightening, we obtain a functor $\mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}\mathrm{at}_\infty$. Composing with the core functor $(-)^{\simeq}\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{K}\mathrm{an}$, we have a functor $S(\mathcal{C})\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{K}\mathrm{an}$.

In practice, one has a more explicit construction, originally due to Waldhausen [17], see also [7, Lecture 18]:

Definition 2.19 *Let P be a partially ordered set, and let $P^{(2)} = \{(i,j) \in P \times P | i \leq j\}$. Let $\mathcal{C}$ be an ∞-category with cofibrations. A P-gapped object of $\mathcal{C}$ is a functor $X\colon \mathrm{N}(P^{(2)}) \to \mathcal{C}$ such that*

- *For every $i \in P$, $X(i,i) = *$;*
- *For $i \leq j \leq k$, the natural map $X(i,j) \to X(i,k)$ is a cofibration;*
- *For $i \leq j \leq k$ in P, the subdiagram*

$$\begin{tikzcd} X(i,j) \arrow[r] \arrow[d] & X(i,k) \arrow[d] \\ X(j,j) = * \arrow[r] & X(j,k) \end{tikzcd}$$

is a pushout square. Denote by $\mathrm{Gap}_P(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathrm{N}(P^{(2)}), \mathcal{C})$ spanned by P-gapped objects.

Example 2.20 $[5]^{(2)}$ *is the category corresponds to the following diagram:*

$$\begin{tikzcd}
* \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
& * \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
& & * \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
& & & * \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
& & & & * \arrow[r] & \bullet \arrow[d] \\
& & & & & *
\end{tikzcd} \tag{2.1}$$

On Barwick's paper [2], after Definition 5.1, he claims that

Proposition 2.21 *The simplicial object $\mathrm{Gap}_{[\bullet]}(\mathcal{C})$ of ∞-categories serves as a straightening of $\mathscr{S}$.*

Therefore we can simply assume that $S_n(\mathcal{C}) = \mathrm{Gap}_{[n]}(\mathcal{C})^{\simeq}$. Unfortunately, I don't have a proof for this fact, or a reference for a proof. In the later parts, we only tacitly assume this.

Definition 2.22 *The K-Theory space $K(\mathcal{C})$ of $\mathcal{C}$ is the loop space of the geometric realization of $S(\mathcal{C})$, namely $\Omega|S_\bullet(\mathcal{C})| = \Omega\,\mathrm{hocolim}_{m\in\Delta^{\mathrm{op}}} S_m(\mathcal{C})$.*

We first claim that $K(\mathcal{C})$ satisfies our prescription:

Proposition 2.23

$$\pi_0(K(\mathcal{C})) = K_0(\mathcal{C}).$$

Proof We compute by Waldhausen's construction: $\pi_0(K(\mathcal{C})) = \pi_1|\mathrm{Gap}_{[\bullet]}(\mathcal{C})^{\simeq}| = \pi_1\left(\bigsqcup_m \mathrm{Gap}_{[m]}(\mathcal{C})^{\simeq} \times \Delta^m/\sim\right)$. The fundamental group is determined by the skeleton at $m \leq 2$, and note that $\mathrm{Gap}_{[0]}(\mathcal{C})^{\simeq}$ is a singleton, $\mathrm{Gap}_{[1]}(\mathcal{C})^{\simeq} \simeq \mathcal{C}^{\simeq}$ gives generators, where each object

$$\begin{tikzcd} * \arrow[r] & X \arrow[d] \\ & * \end{tikzcd}$$

in $\mathrm{Gap}_{[1]}(\mathcal{C})^{\simeq}$ "one-to-one" (∞-categorically, namely, the space of which is weakly equivalent to) corresponds to $X \in \mathcal{C}$, and each object

$$\begin{tikzcd} * \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[d] \\ & * \arrow[r] & X'' \arrow[d] \\ & & * \end{tikzcd} \tag{2.2}$$

in $\mathrm{Gap}_{[2]}(\mathcal{C})^{\simeq}$ "one-to-one" corresponds to a cofiber sequence $X' \to X \to X''$, therefore such a 2-cell gives a relation like $[X] = [X'] + [X'']$. The nuance between groups and abelian groups is resolved by Remark 2.10. $\square$

Chapter 3

Exact ∞-categories

Roughly speaking, an exact ∞-category is a "stable" ∞-category with cofibrations and fibrations: a square of a special kind is a pushout square if and only if it is a pullback square. Especially, for Waldhausen's $S_\bullet$-construction, each square is bicartesian. This enables us to simplify the $S_\bullet$-construction, the result of which is a generalization of Quillen's Q-construction. A reference for this chapter is [1].

Definition 3.1 ([3, Definition 2.1]) *An ∞-category $\mathcal{C}$ is preadditive if it is pointed, admits finite products and finite coproducts, and the canonical map $C_1 \sqcup C_2 \to C_1 \times C_2$ is an equivalence for any $C_1, C_2 \in \mathcal{C}$. In this case, we denote $C_1 \oplus C_2 := C_1 \sqcup C_2$.*

Remark 3.2 *In a preadditive ∞-category $\mathcal{C}$, we usually denote the zero object by 0, instead of $*$.*

For a preadditive ∞-category $\mathcal{C}$, and $A \in \mathcal{C}$, there is a fold map $\nabla \colon A \oplus A \simeq A \sqcup A \xrightarrow{(1_A, 1_A)\cdot} A$ and a shear map $A \oplus A \xrightarrow{p_1 \times \nabla} A \oplus A$.

Definition 3.3 ([3, Definition 2.6]) *A preadditive ∞-category $\mathcal{C}$ is additive if and only if for any $A \in \mathcal{C}$, the shear map $A \oplus A \to A \oplus A$ is an equivalence.*

Proposition 3.4 ([3, Proposition 2.3, 2.8]) *An ∞-category $\mathcal{C}$ is (pre)additive if and only if the homotopy category $\mathrm{h}\mathcal{C}$ is (pre)additive.*

Example 3.5 ([3, Example 2.7]) *By Proposition 3.4, an ordinary category C is additive if and only if $\mathrm{N}(C)$ is additive. Moreover, if an ∞-category $\mathcal{C}$ is additive, then ∞-categories equivalent to $\mathcal{C}$ are also additive, and for any simplicial set K, the functor category $\mathrm{Fun}(K, \mathcal{C})$ is additive.*

It follows from Proposition 3.4 that

Proposition 3.6 ([8, Lemma 1.1.2.10]) *Any stable ∞-category is additive.*

Definition 3.7 ([1, Definition 1.3]) *An exact ∞-category is an additive ∞-category $\mathcal{C}$ along with two collections of morphisms, called cofibrations and fibrations, respectively, such that*

- *The ∞-category $\mathcal{C}$ along with cofibrations constitutes an ∞-category with cofibrations;*
- *The ∞-category $\mathcal{C}^{\mathrm{op}}$ along with (opposites of) fibrations constitutes an ∞-category with cofibrations;*
- *For all diagrams*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ Z \arrow[r, "f'"] & W \end{tikzcd} \tag{3.1}$$

  such that f, f' are cofibrations and g, g' are fibrations, it is a pushout square if and only if it is a pullback diagram. In this case, we call the diagram is ambigressive.

Remark 3.8 *In the diagram 3.1, if it is a pushout square, f is a cofibration, and g is a fibration, then f' is a cofibration and g' is a fibration, and therefore it is ambigressive. Dually, if it is a pullback diagram, f' is a cofibration, and g' is a fibration, then f is a cofibration and g is a fibration, and therefore it is ambigressive.*

Definition 3.9 *Given an exact ∞-category $\mathcal{C}$. An exact sequence is a diagram (or a triangle) $X' \xrightarrow{i} X \xrightarrow{p} X''$ being a part of a pushout (or in fact equivalently, pullback) diagram*

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\ * \arrow[r] & X'' \end{tikzcd}$$

where i is a cofibration and p is a fibration (therefore by the definition of an exact ∞-category, it is ambigressive.). In this case, p (or more imprecisely, X'') is called the cofiber of i and i is called the fiber of p.

Remark 3.10 *We see that in this case, an exact sequence is both a fiber sequence and a cofiber sequence. Further more, if $M' \xrightarrow{i} M \xrightarrow{p} M''$ is a cofiber sequence and i is cofibration, then $M' \xrightarrow{i} M \xrightarrow{p} M''$ is an exact sequence, as a consequence of Remark 3.8. Dually, if $M' \xrightarrow{i} M \xrightarrow{p} M''$ is a fiber sequence and p is a fibration, then $M' \xrightarrow{i} M \xrightarrow{p} M''$ is an exact sequence.*

It follows from [6, Appendix A] that

Proposition 3.11 *For an exact category $(\mathcal{C}, \mathcal{E})$, $\mathrm{N}(\mathcal{C})$ along with inflations as cofibrations and deflations as fibrations, constitutes an exact ∞-category. On the other* *hand, if $\mathcal{C}$ is an ordinary category such that $\mathrm{N}(\mathcal{C})$ could be made into an exact ∞-category, then $\mathcal{C}$ has a structure of an exact category $(\mathcal{C}, \mathcal{E})$ with $\mathcal{E}$ being all exact sequences (see Definition 3.9) in $\mathcal{C}$.*

For sake of completeness, we introduce Quillen's original definition of exact categories:

Definition 3.12 (Quillen, see [11, p. 16]) *An exact category $(\mathcal{C}, \mathcal{E})$ (or by abuse of notation, $\mathcal{C}$), is given by*

1. *An additive category $\mathcal{C}$;*
2. *A collection $\mathcal{E}$ of (short) exact sequences $0 \to M' \to M \to M'' \to 0$ (i.e. $M' \to M$ is a kernel of $M \to M''$, and $M \to M''$ is a cokernel of $M' \to M$) in $\mathcal{C}$. All such $M' \to M$ are called inflations, and all such $M \to M''$ are called deflations.*

such that

- *Canonical exact sequences $(0 \to M' \to M' \oplus M'' \to M'' \to 0) \in \mathcal{E}$;*
- *The class of deflations is closed under composition and pullback along arbitrary maps; Dually, the class of inflations is closed under composition and pushout along arbitrary maps;*
- *For any map $M \to M''$ in $\mathcal{C}$ which admits a kernel, and a map $N \to M$ such that the composition $N \to M \to M''$ is a deflation, then $M \to M''$ is also a deflation; dually, for any map $M' \to M$ which admits a cokernel, and a map $M \to N$ such that the composition $M' \to M \to N$ is an inflation, then $M' \to M$ is also an inflation.*

Remark 3.13 *Quillen originally called inflations admissible monmorphisms, and deflations admissible epimorphisms. We adopt the terminology in [6, Appendix A].*

Remark 3.14 *In the [6, Appendix A], one can see that the last axiom of the original definition of exact categories, Definition 3.12 is redundant.*

Example 3.15 *Any stable ∞-category is an exact ∞-category, where every morphism is a cofibration and a fibration.*

Definition 3.16 *A full subcategory $\mathcal{D}$ of an additive category $\mathcal{C}$ is called an additive subcategory if it is closed under equivalences and direct summands.*

Example 3.17 *Given an exact ∞-category $\mathcal{C}$ and a full additive subcategory $\mathcal{D} \subseteq \mathcal{C}$ closed under extension. Then $\mathcal{D}$ could be made into an exact ∞-category:*

- *Every morphism in $\mathcal{D}$ is a cofibration if it is a cofibration in $\mathcal{C}$ and it admits a fiber (as an object) in $\mathcal{D}$;*
- *Every morphism in $\mathcal{D}$ is a fibration if it is a fibration in $\mathcal{C}$ and it admits a cofiber (as an object) in $\mathcal{D}$.*

Fix an exact ∞-category $\mathcal{C}$. Now let's define the Q-construction of $\mathcal{C}$. One is suggested to glimpse the definition of edgewise subdivision, Definition A.21.

Definition 3.18 *The Q-construction $Q(\mathcal{C})$ is defined as a simplicial set: $\Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, $Q(\mathcal{C})_n$ is the collection of functors $\varepsilon^*(\Delta^n)^{\mathrm{op}} \to \mathcal{C}$ from the opposite of the edgewise subdivision of Δ^n to the ∞-category $\mathcal{C}$, such that the image of each square (see diagram A.1 visually for the concept of a square) is ambigressive.*

Proposition 3.19 *$Q(\mathcal{C})$ is an ∞-category.*

Proof $\varepsilon^*(\Lambda^n_k)^{\mathrm{op}} \hookrightarrow \varepsilon^*(\Delta^n)^{\mathrm{op}}$ is an inner anodyne (a combinatorial fact), therefore has the left lifting property with respect to $\mathcal{C} \to *$ since $\mathcal{C}$ is an ∞-category. $\square$

We need to show that

Theorem 3.20 *$K(\mathcal{C}) \simeq \Omega Q(\mathcal{C})$ as Kan complexes.*

To prove this, we construct an auxiliary simplicial Kan complex $Q_\bullet(\mathcal{C})$:

Definition 3.21 *The Kan complex $Q_n(\mathcal{C})$, is defined by the full subcategory of the core of a mapping space, $\mathrm{Hom}(\varepsilon^*(\Delta^n)^{\mathrm{op}}, \mathcal{C})^{\simeq}$, spanned by functors in $Q(\mathcal{C})_n$. In other words, $Q(\mathcal{C})_n$ is the set of vertices in $Q_n(\mathcal{C})$.*

We first note that

Proposition 3.22 ([1, Proposition 3.4]) *The simplicial Kan complex $Q_\bullet(\mathcal{C})$ forms a complete Segal space.*

where the definition of complete Segal spaces is found in [14, Section 4, 6].

It follows from two Quillen equivalences between complete Segal spaces [5, Section 4] that the geometric realization $|Q_\bullet(\mathcal{C})| \simeq Q(\mathcal{C})$. Therefore we are left to show that

Lemma 3.23

$$K(\mathcal{C}) \simeq \Omega|Q_\bullet(\mathcal{C})|$$

Proof It follows from the following observations:

1. $K(\mathcal{C}) \simeq \Omega|\mathrm{Gap}_{[\bullet]}(\mathcal{C})^{\simeq}|$ as Kan complexes by Waldhausen's construction;
2. $|\mathrm{Gap}_{[\bullet]}(\mathcal{C})^{\simeq}| \simeq |(\mathrm{Gap}_{[2n+1]}(\mathcal{C})^{\simeq})_n|$ by Proposition A.23;
3. The subdiagram $\varepsilon^*(\Delta^n)^{\mathrm{op}} \to \mathrm{N}([2n+1]^{(2)})$ (see Example 3.24 for $n = 2$) induces a functor $\mathrm{Fun}(\mathrm{N}([2n+1]^2), \mathcal{C}) \to \mathrm{Fun}(\varepsilon^*(\Delta^n)^{\mathrm{op}}, \mathcal{C})$;

4. When restricting the functor above to functors defined in Gap-construction, the images are ambigressive; This induces a simplicial map $\mathrm{Gap}_{[2n+1]}(\mathcal{C})^{\simeq} \to Q_n(\mathcal{C})$ between Kan complexes by taking the core;

5. The simplicial map above is a weak equivalence of Kan complexes, by virtue of existence and essential uniqueness of limits. □

Example 3.24 *Here is an example for the subdiagram $\varepsilon^*(\Delta^n)^{\mathrm{op}} \to \mathrm{N}([2n+1]^{(2)})$ for $n = 2$, see also Diagram A.1 and 2.1.*

$$\begin{tikzcd}
0 \arrow[r] & ? \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & 02 \arrow[r] \arrow[d] & 01 \arrow[r] \arrow[d] & 00 \arrow[d] \\
 & 0 \arrow[r] & ? \arrow[r] \arrow[d] & 12 \arrow[r] \arrow[d] & 11 \arrow[r] \arrow[d] & ? \arrow[d] \\
 & & 0 \arrow[r] & 22 \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & ? \arrow[d] \\
 & & & 0 \arrow[r] & ? \arrow[r] \arrow[d] & ? \arrow[d] \\
 & & & & 0 \arrow[r] & ? \arrow[d] \\
 & & & & & 0
\end{tikzcd}$$

Chapter 4

$\mathbb{E}_\infty$-monoids and Symmetric Monoidal ∞-categories

The motivation of $\mathbb{E}_\infty$-monoids, roughly speaking, is the same as the one described in Remark 2.12. Especially, it's a generalization of the concept of monoids, where "multiplication" is homotopy-coherently well-defined, homotopy-coherently commutative and homotopy-coherently associative. Symmetric monoidal ∞-categories furnish a rich resource of examples of $\mathbb{E}_\infty$-monoids in $\mathcal{K}$an, the ∞-category of Kan complexes. Roughly speaking, a symmetric monoidal ∞-category is an ∞-category endowed with a multiplication $\otimes$ which is homotopy-coherently commutative, homotopy-coherently associative and an identity. A reference for this chapter is [3]. Further discussions could be found in Lurie's [8].

Given an ∞-category $\mathcal{C}$ with finite products,

Definition 4.1 *An $\mathbb{E}_\infty$-monoid M in $\mathcal{C}$ is a functor $M\colon \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}$ such that for each $n \geq 0$, the morphisms $M(\langle n\rangle) \to M(\langle 1\rangle)$ induced by the inert maps $(\rho^i\colon \langle n\rangle \to \langle 1\rangle)_{1\leq i\leq n}$, where $\rho^i\colon \langle n\rangle \to \langle 1\rangle, i \mapsto 1$, and for each $j \neq i$, $\rho(j) = *$, exhibits $M(\langle n\rangle)$ as an n-fold power of $M(\langle 1\rangle)$ in $\mathcal{C}$. We will sometimes abuse notation and use the same M for the underlying object $M(\langle 1\rangle)$. Denote by $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathcal{F}\mathrm{in}_*), \mathcal{C})$ spanned by $\mathbb{E}_\infty$-monoids.*

We have a natural "multiplication" $m\colon M \times M \to M$ defined by $M \times M \xrightarrow{\sim} M(\langle 2\rangle) \to M$ where the later morphism is induced by $\langle 2\rangle \to \langle 1\rangle, * \mapsto *, 1 \mapsto 1, 2 \mapsto 1$.

Remark 4.2 *The concept of $\mathbb{E}_\infty$-monoid is a counterpart of the concept of monoid in homotopy theory. The adjective "$\mathbb{E}_\infty$-" characterizes homotopy coherent commutativity and homotopy coherent associativity. General concepts such as operads, $\mathcal{O}$-algebras and $\mathcal{O}$-monoids are introduced by Jacob Lurie, [8].*

Proposition 4.3 ([3, Proposition 1.1]) *Let M be an $\mathbb{E}_\infty$-monoid in $\mathcal{C}$. Then the followings are equivalent:*

1. *M admits an inverse map $i\colon M \to M$, i.e., the composition*
$$M \xrightarrow{\Delta} M \times M \xrightarrow{\mathrm{id}\times i} M \times M \xrightarrow{m} M$$
is homotopic to the identity;
2. *The commutative monoid object underlying M in $\mathrm{h}\mathcal{C}$ is a group object;*
3. *The shear map $s\colon M \times M \xrightarrow{p_1 \times m} M \times M$ is an equivalence.*

In this case, we call M an $\mathbb{E}_\infty$-group and denote by $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C}) \subseteq \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ the full subcategory of $\mathbb{E}_\infty$-groups.

Given that $\mathcal{C}$ is presentable, it follows [3, Corollary 4.4] from adjoint functor theorem that

Proposition 4.4 *The inclusion functor $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C}) \hookrightarrow \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ admits a left adjoint $(-)^{\mathrm{gp}}\colon \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{C})$, the group completion functor.*

We also need an explicit description of the group completion, see [15, Section 2]:

Definition 4.5 *Given an $\mathbb{E}_\infty$-monoid $X\colon \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}$. The geometric realization of X, denoted by $|X|$, is the geometric realization of the simplicial object $\mathrm{N}(\varphi) \circ X\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}$, where the functor $\varphi\colon \Delta^{\mathrm{op}} \to \mathcal{F}\mathrm{in}_*$ is defined by $\varphi([n]) = \langle n\rangle$, and for an increasing map $\alpha\colon [k] \to [n]$ in Δ, $\varphi(\alpha)\colon \langle n\rangle \to \langle k\rangle$, if there is an i such that $\alpha(i-1) < j \leq \alpha(i)$, then $\varphi(\alpha)(j) = i$, else $\varphi(\alpha)(j) = *$.*

Proposition 4.6 *Given an $\mathbb{E}_\infty$-monoid $X\colon \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}$, we form $BX\colon \mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}$ by $BX(\langle n\rangle) = |X(\langle n\rangle \times -)|$. Especially, the underlying space $BX(\langle 1\rangle)$ of BX is just the geometric realization of X. The underlying space of the group completion is equivalent to $\Omega BX(\langle 1\rangle)$. Furthermore, $(B^n X)_{n\geq 1}(\langle 1\rangle)$ form a delooping of $BX(\langle 1\rangle)$.*

We are interested in the special case $\mathcal{C} = \mathcal{K}\mathrm{an}$, which we implicitly assume in the later work, of which a bunch of examples come from symmetric monoidal ∞-categories:

Definition 4.7 ([8, Definition 2.0.0.7]) *A symmetric monoidal ∞-category is a coCartesian fibration of simplicial sets $p\colon \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ such that for each $n \geq 0$, the inert maps $(\rho^i\colon \langle n\rangle \to \langle 1\rangle)_{1\leq i\leq n}$ exhibits $\mathcal{C}^{\otimes}_{\langle n\rangle}$ as an n-times power $(\mathcal{C}^{\otimes}_{\langle 1\rangle})^n$.*

The concept of p-coCartesian lift has a counterpart for ordinary categories:

Definition 4.8 *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories and $f\colon c_1 \to c_2$ be a morphism in $\mathcal{C}$ with image $p(f) = \alpha\colon d_1 \to d_2$. The morphism f is p-coCartesian* *or a p-coCartesian lift of α if it has the following property: For every $h\colon c_1 \to c_3$ in $\mathcal{C}$ with image $\gamma = p(h)\colon d_1 \to d_2$ and every $\beta\colon d_2 \to d_3$ such that $\gamma = \beta \circ \alpha$ there is a unique $g\colon c_2 \to c_3$ in $\mathcal{C}$ such that $\beta = p(g)$ and $h = g \circ f$.*

Definition 4.9 *A functor $p\colon \mathcal{C} \to \mathcal{D}$ is a Grothendieck opfibration if for all $c_1 \in \mathcal{C}$ and for all morphisms α in $\mathcal{D}$ with domain $p(c_1)$, there is a p-coCartesian lift $f\colon c_1 \to c_2$ of α.*

Example 4.10 ([8, Construction 2.0.0.1]) *Given a symmetric monoidal category $\mathcal{C}$, we can construct a symmetric monoidal ∞-category $\mathrm{N}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ by taking the nerve to a Grothendieck opfibration of ordinary categories $\mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$, which is defined by*

- *Objects: $\mathrm{Ob}(\mathcal{C}^{\otimes}) = \bigsqcup_{n\geq 0} \mathcal{C}^n$. We will denote an object by $[C_1, \ldots, C_n]$;*
- *Morphisms: A morphism between $[C_1, \ldots, C_n]$ and $[C_1', \ldots, C_m']$ is given by a map $\alpha \in \mathcal{F}\mathrm{in}_*(\langle n\rangle, \langle m\rangle)$ and a collection of morphisms $(f_j \in \mathcal{C}(\bigotimes_{\alpha(i)=j} C_i, C_j'))_{1\leq j\leq m}$;*
- *Compositions are naturally defined.*

Example 4.11 *As a special case of Example 4.10, any commutative monoid, viewed as a discrete category, could be viewed as a symmetric monoidal category, hence we can associate an $\mathbb{E}_\infty$-monoid. See also Remark 4.2. Explicitly, given a commutative monoid M, we can associate an $\mathbb{E}_\infty$-monoid $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}\mathrm{at}_\infty$ by taking the nerve of the functor $X_M\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}$, where $X_M(\langle n\rangle)$ is the discrete category M^n, and for $\alpha \in \mathcal{F}\mathrm{in}_*(\langle m\rangle, \langle n\rangle)$, we set $X_M(\alpha)(x_1, \ldots, x_m) = (y_1, \ldots, y_n)$ where $y_j = \prod_{\alpha(i)=j} x_i$.*

One notes the similarity between the definition of a symmetric monoidal ∞-category and an $\mathbb{E}_\infty$-monoid. In fact, composing the straightening (see Remark 2.14) $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}\mathrm{at}_\infty$ of a symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ with the core functor $\mathcal{C}\mathrm{at}_\infty \to \mathcal{K}\mathrm{an}$, one obtains an associated $\mathbb{E}_\infty$-monoid in $\mathcal{K}\mathrm{an}$.

Another important example of symmetric monoidal ∞-category comes from operations on an ∞-category: Let $\mathcal{C}$ be an ∞-category with finite coproducts. Roughly speaking, the coproduct operation $\sqcup\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is commutative and associative up to a coherent equivalence. To be more precise, one needs to construct a symmetric monoidal ∞-category $\mathcal{C}^{\sqcup}$ whose underlying space (the subcategory over $\langle 1\rangle \in \mathrm{N}(\mathcal{F}\mathrm{in}_*)$) is $\mathcal{C}$ and whose “multiplication” coincides with $\sqcup$ in $\mathcal{C}$. This is called a coCartesian symmetric monoidal structure. We describe the construction, and for technical details, we refer to [8, Section 2.4.3].

Definition 4.12 ([8, Construction 2.4.3.1]) *We define a category Γ^* as follows:*

- *The objects of Γ^* are pairs $(\langle n\rangle, i)$ where $* \neq i \in \langle n\rangle^\circ$;*

- *A morphism in Γ^* from $(\langle m\rangle, i)$ to $(\langle n\rangle, j)$ is a map $\alpha \in \mathcal{F}\mathrm{in}_*(\langle m\rangle, \langle n\rangle)$ such that $\alpha(i) = j$.*

Definition 4.13 *We define the simplicial set $\mathcal{C}^\sqcup$ equipped with a map $\mathcal{C}^\sqcup \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ by the following universal property: for every simplicial map $K \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$, we have a bijection*

$$\mathrm{Hom}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(K, \mathcal{C}^\sqcup) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(K \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^*), \mathcal{C})$$

Take $K \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ to be constant functors, we obtain that the fiber of $\mathcal{C}^\sqcup \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ over $\langle n\rangle$ could be identified with $\mathcal{C}^n$, similar to the construction of Example 4.10.

Proposition 4.14 ([8, Remark 2.4.3.4]) *As defined above, $\mathcal{C}^\sqcup \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a symmetric monoidal ∞-category.*

As a result of these preparations of symmetric monoidal ∞-categories and $\mathbb{E}_\infty$-monoids, we now define the additive K-theory space for an ∞-category with finite coproducts $\mathcal{C}$. We note that, the $\mathbb{E}_\infty$-monoid in $\mathcal{K}$an associated to the coCartesian symmetric monoidal structure on $\mathcal{C}$ has an underlying space $\mathcal{C}^\simeq$, hence by abuse of notation, we also denote by $\mathcal{C}^\simeq$ the $\mathbb{E}_\infty$-monoid.

Definition 4.15 *The additive K-theory space of $\mathcal{C}$, $K_{\mathrm{add}}(\mathcal{C})$, is defined as the underlying space of the group completion $(\mathcal{C}^\simeq)^{\mathrm{gp}}$. By abuse of notation, we also denote by $K_{\mathrm{add}}(\mathcal{C})$ the group completion itself.*

Chapter 5

A Comparison Theorem and a Cofinality Theorem

This chapter is devoted to the proof of a cofinality theorem for additive K-Theory, and the main comparison theorem of the exact K-theory and the additive K-theory.

Proposition 5.1 (Cofinality for Additive K-Theory, [7, Lecture 18, Prop 4]) *Let $\mathcal{C}$ be an additive ∞-category and let $\mathcal{C}_0$ be a full subcategory such that every object of $\mathcal{C}$ is a direct summand of an object of $\mathcal{C}_0$. Then the diagram*

$$\begin{array}{ccc} K_{\text{add}}(\mathcal{C}_0) & \longrightarrow & K_{\text{add}}(\mathcal{C}) \\ \big\downarrow & & \big\downarrow \\ \pi_0(K_{\text{add}}(\mathcal{C}_0)) & \longrightarrow & \pi_0(K_{\text{add}}(\mathcal{C})) \end{array}$$

is a pullback square in $\mathcal{K}\text{an}$, where π_0-groups are endowed with discrete topologies.

Proof We first give an alternative description of $\pi_0(K_{\text{add}}(\mathcal{C}))$ (and a similar one for $\mathcal{C}_0$). In fact, it is just the free (abelian) group generated by elements in $\mathcal{C}$, modulo the relations $[X' \oplus X''] = [X'] + [X'']$ for $X', X'' \in \mathcal{C}$. This implies that, X, X' have the same image in $\pi_0(K_{\text{add}}(\mathcal{C}_0))$ if and only if there exists $Y \in \mathcal{C}_0$ such that $X \oplus Y \simeq X' \oplus Y$. Since for $Z \in \mathcal{C}$, there exists $Z' \in \mathcal{C}$ such that $Z \oplus Z' \in \mathcal{C}_0$, the previous condition is equivalent to the existance of $Z \in \mathcal{C}$ such that $X \oplus Z \simeq X' \oplus Z$, which is equivalent to say that X, X' have the same image in $\pi_0(K_{\text{add}}(\mathcal{C}))$. As a consequence, the bottom arrow is injective.

Since all spaces in question are simple, it suffices to show that the square is a pullback square after passage to singular (or simplicial) chain complexes with coefficient $\mathbb{Z}$.

Let $A_0 = C_*(K_{\text{add}}(\mathcal{C}_0)), A = C_*(K_{\text{add}}(\mathcal{C}))$, and $B_0 = C_*(\mathcal{C}_0^{\simeq}), B = C_*(\mathcal{C}^{\simeq})$, all of which admit an $\mathbb{E}_\infty$-algebra structure. We note that there is a $\mathcal{C}_0^{\simeq}$-action on B_0 by multiplication by elements of the form $[X] \in H_0(B_0)$ for $X \in \mathcal{C}_0$, and a $\mathcal{C}^{\simeq}$-action on B. By properties of group completion, A_0, A are obtained by inverting the actions on B_0, B, respectively. However, since every object of $\mathcal{C}$ is a direct summand of an object in $\mathcal{C}_0$, we only need to invert $\mathcal{C}_0^{\simeq}$-action on B, via the inclusion $\mathcal{C}_0^{\simeq} \subseteq \mathcal{C}^{\simeq}$, which implies that $A \simeq B(A_0, B_0, B)$. We note that B contains B_0 as a direct summand, we deduce the proposition. □

Theorem 5.2 (Comparison, [7, Lecture 18, Thm 10]) *Let $\mathcal{C}$ be an additive ∞-category. Regard $\mathcal{C}$ as an ∞-category with cofibrations as in Example 2.6 (allowing only split cofibrations). Then there is a canonical homotopy equivalence $K_{\text{add}}(\mathcal{C}) \to K(\mathcal{C})$.*

Proof Our plan is the following: first, we construct a functor $\mathcal{C}^{\sqcup} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathscr{S}(\mathcal{C})$ over $\mathrm{N}(\Delta^{\mathrm{op}})$, which induces a functor between simplicial objects $Y_\bullet \to S_\bullet(\mathcal{C})$, where $Y_\bullet$, being a simplicial object in $\mathcal{K}\mathrm{an}$, is the composition of the straightening-then-taking-core of $\mathcal{C}^{\sqcup} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ with $\mathrm{N}(\Delta^{\mathrm{op}}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. Then we need to show that, this leads to an equivalence between geometric realizations (see Definition 2.22 and Proposition 4.6). This will lead to the result.

Roughly speaking, the functor $\mathcal{C}^{\sqcup} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathscr{S}(\mathcal{C})$ is defined by $(C_1, \ldots, C_n) \mapsto (0 \to C_1 \to C_1 \oplus C_2 \to C_1 \oplus C_2 \oplus C_3 \to \cdots \to C_1 \oplus C_2 \oplus \cdots \oplus C_n)$. We need to prescribe the evaluation of the functor on higher dimensional cells naturally. Instead of directly defining the functor, we obtain the functor from a functor $\mathcal{C}^{\sqcup} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathscr{F}(\mathcal{C})$ defined by Yoneda lemma. In light of Definition 4.13 and Proposition 2.17, we need to prescribe a map $\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(K \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Gamma^*), \mathcal{C}) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(K \times_{\mathrm{N}(\Delta^{\mathrm{op}})} \mathrm{N}(M), \mathcal{C})$ natural in $K \to \mathrm{N}(\Delta^{\mathrm{op}})$. This is determined by the expression for objects above.

Next, we need to study the geometric realizations. First, we note that both $\mathbb{E}_\infty$-monoids $|Y_\bullet|$ (defined as the homotopy colimit of the simplicial object in $\mathbb{E}_\infty$-monoids: we identify each space $|Y_\bullet|$ with an $\mathbb{E}_\infty$-monoid, see B-construction in Proposition 4.6, which is equivalent to the one obtained from the coCartesian symmetric monoidal structure) and $|S_\bullet(\mathcal{C})|$ (each space is also an underlying space of an $\mathbb{E}_\infty$-monoid by considering the coCartesian symmetric monoidal structure) are group-like because they are connected, hence are $\mathbb{E}_\infty$-groups, which implies that they can be identified with their group completions respectively. The group completion functor, being a left adjoint, commutes with homotopy colimits, especially, commutes with geometric realizations. Therefore $|Y_\bullet| \simeq |Y_\bullet|^{\mathrm{gp}} \simeq |Y_\bullet^{\mathrm{gp}}|$ and $|S_\bullet(\mathcal{C})| \simeq |S_\bullet(\mathcal{C})|^{\mathrm{gp}} \simeq |S_\bullet(\mathcal{C})^{\mathrm{gp}}|$. Thus it will suffice to show Lemma 5.3. □

Lemma 5.3 *For $n \geq 0$, the map $Y_n \to S_n(\mathcal{C})$, which is equivalent to the core of the fiber of the functor $\mathcal{C}^{\sqcup} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathscr{S}(\mathcal{C})$ over $[n] \in \Delta^{\mathrm{op}}$, i.e. $(\mathcal{C}^n)^{\simeq} \to \mathscr{S}_n(\mathcal{C})^{\simeq}$, induces a homotopy equivalence after group completion.*

Definition 5.4 *Given a morphism $f\colon M \to N$ between two $\mathbb{E}_\infty$-monoids. We view M acting on N: $(m, n) \mapsto m + n$. We define the one-sided bar construction $B(N, M, *)$ to be the homotopy colimit of the functor $\mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{K}\mathrm{an}, [n] \mapsto N \times M_{\langle n-1\rangle} \simeq N \times M_{\langle 1\rangle}^{n-1}$.*

We first admit following lemmas:

Lemma 5.5 *Given a morphism $f\colon M \to N$ between two $\mathbb{E}_\infty$-monoids. The homotopy cofiber of f coincides with the one-sided bar construction $B(N, M, *)$.*

Lemma 5.6 *Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ which is compatible with an M-action where M is an ∞-monoid and M acts trivially on $\mathcal{D}$. F realizes as a bar construction $\mathcal{C} \to B(\mathcal{C}, M, *)$ if and only if for each object $x \in \mathcal{D}$, the bar construction of the fiber of $\mathcal{C}$ over x, $B(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/x}, M, *)$ is contractible. In other words, where a map realizes as a bar construction could be checked pointwisely.*

$\delta_n\colon [n-1] \to [n], x \mapsto x$ induces $e\colon \mathscr{S}_n(\mathcal{C})^{\simeq} \to \mathscr{S}_{n-1}(\mathcal{C})^{\simeq}$. Fix $X = (0 \to X_1 \to \cdots \to X_{n-1}) \in \mathscr{S}_{n-1}(\mathcal{C})^{\simeq}$.

Lemma 5.7 *The fiber of $\mathscr{S}_{n-1}(\mathcal{C})^{\simeq}$ over X is homotopically equivalent to $\mathcal{D}^{\simeq}$, where $\mathcal{D}$ is the full subcategory of $\mathcal{C}_{X_{n-1}/}$ spanned by split cofibrations $X_{n-1} \to X''$.*

Proof (Lemma 5.3) We start with an examination of the following diagram:

$$\begin{array}{ccccc} \mathcal{C}^{\simeq} & \xrightarrow{\eta} & \mathscr{S}_{n-1}(\mathcal{C})^{\simeq} \times \mathcal{C}^{\simeq} & \xrightarrow{f} & \mathscr{S}_n(\mathcal{C})^{\simeq} \\ \downarrow & & \downarrow \pi & & \downarrow e \\ * & \longrightarrow & \mathscr{S}_{n-1}(\mathcal{C})^{\simeq} & \xrightarrow{\mathrm{id}} & \mathscr{S}_{n-1}(\mathcal{C})^{\simeq} \end{array}$$

where $\eta\colon \mathcal{C}^{\simeq} \to \mathscr{S}_{n-1}(\mathcal{C})^{\simeq} \times \mathcal{C}^{\simeq}$ is the inclusion $X \mapsto (*, X)$, $\pi\colon \mathscr{S}_{n-1}(\mathcal{C})^{\simeq} \times \mathcal{C}^{\simeq} \to \mathscr{S}_{n-1}(\mathcal{C})^{\simeq}$ is the projection $(X, Y) \mapsto X$. f is defined by $((0 \to X_1 \to \cdots \to X_{n-1}), Y) \mapsto (0 \to X_1 \to \cdots \to X_{n-1} \to X_{n-1} \oplus Y)$. Our goal is to show that f becomes an equivalence after group completion, from which the proposition follows inductively on n. In other words, we need to show that the square on the right becomes a pullback square after group completion, or a pushout square after group completion, due to the fact that $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{K}\mathrm{an})$ is stable. Clearly, the square on the left becomes a pushout square after group completion, therefore it suffices to show that the outer square becomes a pushout square after group completion. In fact, the outer square is a pushout square before group completion, i.e. e realizes as a homotopy cofiber of $\mathcal{C}^{\simeq} \to \mathscr{S}_n(\mathcal{C})^{\simeq}$. By preceding lemmas, we are left to show Lemma 5.8 $\square$

Lemma 5.8 *Fix $X \in \mathcal{C}$, and let $\mathcal{D}$ denote the full subcategory of $\mathcal{C}_{X/}$ spanned by the split cofibrations $X \to X'$. Then the homotopy cofiber of $f\colon \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}, C \mapsto (X \to X \oplus C)$ is contractible.*

Proof Since each cofibration in $\mathcal{C}$ splits, the homotopy cofiber of f is connected, therefore group-like. Therefore we only need to show that, f becomes an equivalence after group completion.

Let $q\colon \mathcal{D}^{\simeq}, (X \to X') \mapsto X'/X$. It is obvious that $q \circ f$ is homotopically equivalent to id before group completion. To complete the proof, we need to show that $f \circ q\colon \mathcal{D}^{\simeq} \to \mathcal{D}^{\simeq}, (X \to X') \mapsto (X \to X \oplus (X'/X))$ is homotopically equivalent to id after group completion. To show this, we first note that, the "multiplication" (or addition) in $\mathcal{D}^{\simeq}$ is given by $(X \to X') + (X \to X'') = X \to X' \sqcup_X X''$. We "multiply" a simple copy of the identity map id to $f \circ q$ and id. In other words, it suffices to show that we can identify $X \to X' \sqcup_X X'$ and $X \to X' \oplus (X'/X)$ functorially in $X \to X'$. This is given by a natural map $X' \sqcup_X X' \to X' \oplus (X'/X)$. $\square$

Chapter 6

Relations with Classical Constructions

Historically, the first successful higher K-group is obtained from Quillen's $+$-construction:

Definition 6.1 *Let (X, x) be a pointed path-connected space and $N \trianglelefteq \pi_1(X, x)$ is a perfect normal subgroup. The plus construction gives us a pointed path-connected space (X^+, x^+) along with a continuous map $f\colon (X, x) \to (X^+, x^+)$ determined by the following properties:*

1. *The induced morphism $\pi_1(f)\colon \pi_1(X, x) \to \pi_1(X^+, x^+)$ is a surjection, with a kernel isomorphic to N;*
2. *For any local coefficient system L on X^+, $f_*\colon H_n(X, f^*L) \to H_n(X^+, L)$ is an isomorphism for any $n \geq 0$;*
3. *If $g\colon (X, x) \to (Y, y)$ is a continuous map such that $N \subseteq \ker(\pi_1(g))$, then there exists a continuous map $h\colon (X^+, x^+) \to (Y, y)$ such that $h \circ f \simeq g$.*

Quillen showed [16, Theorem 2.1] the existence and the uniqueness (up to homotopy) of (X^+, x^+).

Let R be a unital ring. Denote by $GL(R) = \operatorname{colim}_n GL_n(R)$ the general linear group over R, where inclusion $GL_n(R) \to GL_{n+1}(R)$ is given by $A \mapsto \left[\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Set $E(R) = \operatorname{colim}_n E_n(R)$, where $E_n(R) \subseteq GL_n(R)$ is the subgroup of elementary matrices. $E(R)$ is perfect [16, Proposition 1.5]: $E(R) = [E(R), E(R)] = [GL(R), GL(R)]$. Denote by BG the classifying space (a simplicial set) of a group G. Quillen defined the K-theory space of R to be $K_0(R) \times BGL(R)^+$, the perfect subgroup in question is $E(R)$. We note that the ∞-category with cofibrations $\mathcal{P}(R)$ defined in Example 2.3 coincides with the cofibrations prescribed in Example 2.6, therefore it follows from Theorem 5.2 that $K(\mathcal{P}(R)) \simeq K_{\text{add}}(\mathcal{P}(R))$. Now we will show that

Proposition 6.2

$$K_{\text{add}}(\mathcal{P}(R)) \simeq K_0(R) \times BGL(R)^+$$

We need the following lemma:

Lemma 6.3 *For a simplicial monoid M with identity $* \in M$, the bar construction $B(M, M, *)$ is contractible.*

Proof

Step 1: We start with the special case that M is just a monoid. Denote $E(M)$ the category defined by

- Objects: $\mathrm{Ob}(E(M)) = M$;
- Morphisms: $\mathrm{Hom}(x, y) = \{ z \in M \mid zx = y \}$;
- Compositions: given $u \in \mathrm{Hom}(y, z)$ such that $uy = z$ and $v \in \mathrm{Hom}(x, y)$ such that $vx = y$, then $uvx = z$. We define $u \circ v = uv$.

By definition, $B(M, M, *) = \mathrm{N}(E(M))$. We note that the identity $* \in E(M)$ is the initial object in $E(M)$, hence $\mathrm{N}(E(M))$ is contractible.

Step 2: In general, we need to show that $* \to B(M, M, *)$ is a weak equivalence of simplicial sets. We note that for each $n \in \mathbb{N}$, M_n is a monoid, hence by the previous step, we can retract $B(M_n, M_n, *)$ to the point of the identity in M_n. However, since the identity is just the image of $* \in M_0$ under degenerate maps, therefore we deduce that $* \to B(M, M, *)$ is a weak equivalence of simplicial sets by Theorem A.20. □

Proof (Proposition 6.2) We only need to show that $K_{\mathrm{add}}(\mathcal{P}(R))_0 \simeq BGL(R)^+$, where $K_{\mathrm{add}}(\mathcal{P}(R))_0$ is the connected component of 0. Denote by $\mathcal{F}(R) \subseteq \mathcal{P}(R)$ the full subcategory spanned by finitely generated free R-modules. It follows from Proposition 5.1 that $K_{\mathrm{add}}(\mathcal{P}(R))_0 \simeq K_{\mathrm{add}}(\mathcal{F}(R))_0$. We are left to show that

$$K_{\mathrm{add}}(\mathcal{F}(R))_0 \simeq BGL(R)^+ \tag{6.1}$$

Let's consider $\mathcal{F}(R)^{\simeq}$. Direct sum with R induces a functor $R \oplus - \colon \mathcal{F}(R)^{\simeq} \to \mathcal{F}(R)^{\simeq}$, subsequently forms a direct system $\mathcal{F}(R)^{\simeq} \xrightarrow{R\oplus} \mathcal{F}(R)^{\simeq} \xrightarrow{R\oplus} \dots$. Denote by X the (ordinary) colimit in $\mathcal{S}\mathrm{et}_\Delta$, which could be identified with $\mathbb{Z} \times BGL(R)$. $\mathcal{F}(R)^{\simeq}$ acts on the left of $\mathcal{F}(R)^{\simeq}$, hence on X. It follows from the group completion theorem 6.4 that X is homologically isomorphic to the homotopy fiber of $B(X, \mathcal{F}(R)^{\simeq}, *) \to B(*, \mathcal{F}(R)^{\simeq}, *)$. We note that the one-sided bar construction $B(X, \mathcal{F}(R)^{\simeq}, *)$ is contractible because it is a filtered colimit of contractible spaces $B(\mathcal{F}(R)^{\simeq}, \mathcal{F}(R)^{\simeq}, *)$, thus X is homologically isomorphic to $K_{\mathrm{add}}(\mathcal{F}(R))$.

Since both spaces in 6.1 are simple ($BGL(R)^+$ is an H-space, see [16, Proposition 2.9]) and homologically isomorphic, hence equivalent. □

Theorem 6.4 (Group completion, [10, Ch 4, Thm 5.15]) *Suppose that X is a simplicial set with a right action of a simplicial monoid M. The action of each vertex of M induces isomorphisms on the homology. Then the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & B(X, M, *) \arrow[d] \\ * \arrow[r] & B(*, M, *) \end{tikzcd}$$

is homology cartesian in the sense that the induced morphism from X to the homotopy fiber of $B(X, M, *) \to B(*, M, *)$ induces isomorphisms on homology.

Next we describe Quillen's original Q-construction [11, p. 100]:

Definition 6.5 *Let $\mathcal{C}$ be an exact category (see Definition 3.12). We form a new category $Q\mathcal{C}$, having the same objects as $\mathcal{C}$, with morphisms defined in the following way: Let $M, M' \in \mathcal{C}$ and consider all diagrams $M \overset{p}{\twoheadleftarrow} N \overset{j}{\hookrightarrow} M'$, where p is a deflation and j is an inflation. Given a morphism $M' \to M''$ represented by the diagram $M' \overset{p'}{\twoheadleftarrow} N' \overset{j'}{\hookrightarrow} M''$, the composition of $M' \to M''$ and $M \to M'$ is represented by $M \twoheadleftarrow N \times_{M'} N' \hookrightarrow M''$, where the morphisms are read from the following diagram:*

$$\begin{tikzcd} N \times_{M'} N' \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & M'' \\ N \arrow[r] \arrow[d] & M' & \\ M & & \end{tikzcd}$$

We claim that Barwick's Q-construction, Definition 3.18, coincides with Quillen's Q-construction 6.5 when $\mathcal{C}$ comes from an ordinary exact category:

Proposition 6.6 *Given an ordinary exact category $(\mathcal{C}, \mathcal{E})$. It follows from Proposition 3.11 that $\mathrm{N}(\mathcal{C})$ is an exact ∞-category. Moreover, we have $Q(\mathrm{N}(\mathcal{C}))$ is categorically equivalent to $\mathrm{N}(Q\mathcal{C})$.*

Proof

Step 1: We'd show that, the space of right morphisms $\mathrm{Hom}^{\mathrm{R}}_{Q(\mathrm{N}(\mathcal{C}))}(X, Y)$, as a Kan complex, is weak equivalent to the nerve of the groupoid $G(X, Y)$ of which an object is $X \overset{p}{\twoheadleftarrow} Z \overset{j}{\hookrightarrow} Y$, where p is a deflation and j is an inflation, the component in the definition of Quilen's Q-construction $Q\mathcal{C}$, and morphisms are just isomorphisms between these components. Unwinding the definitions, an n-simplex of $\mathrm{Hom}^{\mathrm{R}}_{Q(\mathrm{N}(\mathcal{C}))}(X, Y)$ looks like (for $n = 3$):

$$\begin{tikzcd}
Z_1 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & Z_3 \arrow[r] \arrow[d] & Z_4 \arrow[r] \arrow[d] & Y \\
X \arrow[r, "="] \arrow[d, "="'] & X \arrow[r, "="] \arrow[d, "="'] & X \arrow[r, "="] \arrow[d, "="'] & X & \\
X \arrow[r, "="] \arrow[d, "="'] & X \arrow[r, "="] \arrow[d, "="'] & X & & \\
X \arrow[r, "="] \arrow[d, "="'] & X & & & \\
X & & & &
\end{tikzcd}$$

Since each square is ambigressive, the arrows except the rightmost one (in the example above, they are $Z_1 \xrightarrow{\sim} Z_2 \xrightarrow{\sim} Z_3 \xrightarrow{\sim} Z_4$) in the first row are all isomorphisms, therefore the space of right morphisms is isomorphic to the nerve of the groupoid $G(X, Y)$. For $n = 3$, the diagram above corresponds to a 3-cell in $G(X, Y)$, whose vertices are $X \twoheadleftarrow Z_1 \to Y$, $X \twoheadleftarrow Z_2 \to Y$, $X \twoheadleftarrow Z_3 \to Y$, and faces are induced by the isomorphisms on the top line.

Step 2: It follows from the fact that j is a monomorphism that there is at most one (iso)morphism between two objects in $G(X, Y)$, therefore each connected component of $\mathrm{N}(G(X, Y))$ is contractible, from which we deduce that $Q(\mathrm{N}(\mathcal{C}))$ is categorically equivalent to $\mathrm{N}(Q\mathcal{C})$, by [9, Proposition 2.3.4.18], a characterization of an n-category via mapping spaces. □

Appendix A

Simplicial Sets

A general reference for simplicial sets is [10].

Definition A.1 *The category Δ is the category of finite ordinal numbers with order-preserving maps. To be more precise, let $[n] = \{0, 1, \dots, n\}$ be the totally ordered set with $n + 1$ elements, then $\mathrm{Ob}(\Delta) = \{[n] \mid n \in \mathbb{N}\}$, and a morphism from $[m]$ to $[n]$ is a non-decreasing map $[m] \to [n]$.*

There is a natural "model" of Δ in $\mathcal{T}\mathrm{op}$, namely a functor $\|\cdot\| \colon \Delta \to \mathcal{T}\mathrm{op}, [n] \mapsto \|\Delta^n\|$, where $\|\Delta^n\|$ is the standard n-simplex in $\mathcal{T}\mathrm{op}$: $\{(x_0, \dots, x_n) \in \mathbb{R}^n \mid x_0 + \cdots + x_n = 1\}$. Given a morphism $\alpha \colon [m] \to [n]$ in Δ, the corresponding $\|[m]\| \to \|[n]\|$ is defined by the linear extension of $(e_j \mapsto e_{\alpha(j)})_{0 \leq j \leq m}$.

Example A.2 *Let X be a topological space. The singular set $S(X)$ is a simplicial set defined by $S(X)([n]) = \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(\|\Delta^n\|, X)$. This will define a functor $S \colon \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}_\Delta$.*

Example A.3 *Let $\mathcal{C}$ be a category. Since every partially ordered set could be seen as a small category, we associate a simplicial set $\mathrm{N}(\mathcal{C})$ to $\mathcal{C}$, called the nerve of the category $\mathcal{C}$, defined by $\mathrm{N}(\mathcal{C})([n]) = \mathrm{Fun}([n], \mathcal{C})$. Especially, we can associate any group G a category with only one object canonically. We denote by BG the nerve of the category in question, and usually call it the classifying space of G.*

Definition A.4 *A simplicial set is a functor $X \colon \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, where $\mathcal{S}\mathrm{et}$ is the category of sets.*

Example A.5 *Denote by $\Delta^n \colon \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}, [m] \to \mathrm{Hom}_\Delta([m], [n])$ the representable functor. We call Δ^n the n-simplex.*

Definition A.6 *A simplicial map between two simplicial sets is a natural transformation between two functors. Denote by $\mathcal{S}\mathrm{et}_\Delta = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ the category of simplicial sets where morphisms are simplicial maps.*

As a presheaf category, we have

Proposition A.7 *The category $\mathcal{S}\mathrm{et}_\Delta$ of simplicial sets is complete and cocomplete, especially admits products, coproducts, pushouts and pullbacks. The limits and colimits are computed pointwisely.*

Definition A.8 *An n-simplex of a simplicial set $X\colon \Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is an element $x_n \in X([n])$, one-to-one corresponding to a simplicial map $\Delta^n \to X$ by Yoneda lemma. x_n is called degenerate if there is an $n-1$-face $y_{n-1} \in X([n+1])$ and $\alpha \in \Delta([n],[n-1])$ such that $X(\alpha^{\mathrm{op}})(y_{n-1}) = x_n$. x_n is called non-degenerate if it is not degenerate.*

Example A.9 *Fix $n \in \mathbb{N}$. There are $n+1$ injections $[n-1] \to [n]$: $\delta_j(i) = i$ if $i < j$, and $\delta_j(i) = i+1$ if $i \geq j$, for $0 \leq j \leq n$. These induces $n+1$ simplicial maps $\delta_j\colon \Delta^{n-1} \to \Delta^n$ (called j-th face of Δ^n). For $k \in \mathbb{N}$, $\delta_j([k])$ is injective. Fix $0 \leq k \leq n$, we form the union of n faces, namely $\Lambda^n_k([m]) = \bigcup_{j \neq k} \delta_j(\Delta^{n-1}([m]))$, and the boundary $\partial\Delta^n([m]) = \bigcup_j \delta_j(\Delta^{n-1}([m]))$. Λ^n_k and $\partial\Delta^n$ as subfunctors of Δ^n, form simplicial sets, called the k-th horn of Δ^n, and the boundary of Δ^n, respectively. Especially, if $k = 0$ or $k = n$, then we say that Λ^n_k is an outer horn, otherwise we say that Λ^n_k is an inner horn.*

Definition A.10 *A simplicial set is called finite if it contains only a finite number of non-degenerate simplexes.*

The functor $\|\cdot\|\colon \Delta \to \mathcal{T}\mathrm{op}$ extends along the Yoneda embedding $\Delta \to \mathcal{S}\mathrm{et}_\Delta$ (in fact, a Kan extension), by the following coend expression:

Definition A.11 *Given a simplicial set X, we define the geometric realization of X,*

$$\|X\| = \int^{[n]\in\Delta} X([n]) \times \|\Delta^n\|$$

It follows from abstract nonsense that

Proposition A.12 *The geometric realization functor $\|\cdot\|\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{T}\mathrm{op}$ is a left adjoint to the singular set functor $S\colon \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}_\Delta$.*

The category $\mathcal{S}\mathrm{et}_\Delta$ also admits exponentials:

Definition A.13 *Let X, Y be simplicial sets. The mapping space* $\mathrm{Map}(X,Y)$ *is the simplicial set defined by* $\mathrm{Map}(X,Y)([n]) = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(X \times \Delta^n, Y)$ *with natural choices for* $\mathrm{Map}(X,Y)(\alpha^{\mathrm{op}})$ *for morphisms α in Δ.*

There is a canonical simplicial model structure on $\mathcal{S}\mathrm{et}_\Delta$:

Theorem A.14 ([10, Section I.11]) *The category $\mathcal{S}\mathrm{et}_\Delta$ admits model structure, where cofibrations are pointwise injections and weak equivalences are weak equivalences after geometric realization. With this model structure, $\mathcal{S}\mathrm{et}_\Delta$ form a simplicial model category. The geometric realization functor $\|\cdot\|$ along with the singular set functor S form a Quillen equivalence $\|\cdot\| \dashv S$.*

The fibrations for this model structure are called Kan fibrations, characterized by the following property:

Proposition A.15 *A simplicial map is a Kan fibration if and only if it has right lifting property with respect to all horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$.*

Especially, fibrant simplicial sets for this model structure are called Kan complexes.

Definition A.16 *Let $\mathcal{C}$ be a category. A functor $\Delta^{\mathrm{op}} \to \mathcal{C}$ is called a simplicial object in $\mathcal{C}$. Morphisms between simplicial objects are just natural morphisms, as in the case of simplicial sets. Especially, the simplicial objects in $\mathcal{S}\mathrm{et}_\Delta$ are called bisimplicial sets. The category of bisimplicial sets $\mathcal{S}\mathrm{et}_{\Delta^2}$ is equivalent to* $\mathrm{Fun}(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathcal{S}\mathrm{et})$.

Definition A.17 *The diagonal functor $\Delta\colon \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ induces a functor $d\colon \mathcal{S}\mathrm{et}_{\Delta^2} \to \mathcal{S}\mathrm{et}_\Delta$. The image of a bisimplicial set X is the diagonal simplicial set of X, denoted by $|X|$. By definition, $|X|([n]) = X([n],[n])$.*

As the notation indicates, the diagonal simplicial set could be viewed as a kind of geometric realization:

Proposition A.18 ([10, Exercise II.1.4])

$$|X| = \int^{[n]\in\Delta} X([n],\bullet) \times \Delta^n$$

Definition A.19 *A morphism between bisimplicial sets is called a weak equivalence if it induces a weak equivalence between diagonal simplicial sets.*

Notationally, we usually use the subscript for simplicial sets and bisimplicial sets, say X_n for $X([n])$, etc. As a consequence of Proposition A.18,

Theorem A.20 ([10, Chapter IV, Proposition 1.7, p. 199]) *Given two bisimplicial sets $X_{\bullet,\bullet}, Y_{\bullet,\bullet}$ and a bisimplicial map $f_{\bullet,\bullet}\colon X_{\bullet,\bullet} \to Y_{\bullet,\bullet}$. If for every $[m] \in \Delta^{\mathrm{op}}$, the map $f_{m,\bullet}$ is a weak equivalence, then the induced map $|f|$ between diagonals of X, Y is also a weak equivalence.*

Let's define the edgewise subdivision:

Denote by $\varepsilon\colon \Delta \to \Delta$ the functor given by $[n] \to [n]^{\mathrm{op}} \star [n] \cong [2n+1]$, where $\star$ is the concatenate of two finite totally ordered sets, $[n]^{\mathrm{op}}$ is the opposite category of $[n]$, which is equivalent to $[n]$, but this notation indicates that $\varepsilon(\alpha)$ is the concatenate of $n - \alpha(n - \bullet)$ and α.

Definition A.21 (edgewise subdivision) *Given a simplicial object $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$, we define the edgewise subdivision $\varepsilon^* X$ of X to be $X \circ \varepsilon^{\mathrm{op}}$, namely the composition $\Delta^{\mathrm{op}} \xrightarrow{\varepsilon^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{X} \mathcal{C}$.*

Example A.22 $\varepsilon^*\Delta^2$ *is the nerve of the category determined by the following diagram:*

$$
\begin{tikzcd}
 & & 22 \arrow[d] \\
 & 11 \arrow[r] \arrow[d] & 12 \arrow[d] \\
00 \arrow[r] & 01 \arrow[r] & 02
\end{tikzcd}
\tag{A.1}
$$

Usually edgewise subdivision doesn't "change" the object homotopically:

Proposition A.23 ([1, Section 2]) *If X is a simplicial set, then $\|X\|$ is homeomorphic to $\|\varepsilon^* X\|$. If X is a bisimplicial set, viewed as a simplicial object in $\mathcal{S}\mathrm{et}_\Delta$, then the geometric realizations of $|X|$ and $|\varepsilon^* X|$ are homeomorphic.*

Appendix B

∞-categories

We review some basic concepts of ∞-categories, which is the basic language of this thesis, and a paradigm of homotopy coherency, see Remark 2.12. The axioms of categories find their homotopy counterparts. There are many ways to do this, called different models of ∞-categories. We follow Joyal and Lurie's approach. A bibliographical reference for ∞-categories is Lurie's [9]. A good introduction is Groth's notes [4].

Definition B.1 *A simplicial map $X \to S$ is called an inner fibration if it has right lifting property with respect to all inner horn maps $\Lambda^n_k \hookrightarrow \Delta^n$ for $0 < k < n$.*

Definition B.2 *A simplicial set X is called an ∞-category if the constant map $X \to *$ is an inner fibration.*

In other words, a simplicial set X is an ∞-category if and only if it has extension property for inner horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$.

Example B.3 *Given an ordinary category $\mathcal{C}$. Then the nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category.*

In an ∞-category, vertices are called objects, denoted as $x \in \mathcal{C}$; edges are called morphisms, denoted as $f\colon x \to y$.

Example B.4 *There is a canonical choice of an edge for any objects: the constant map (or called the degeneracy map) $\Delta^1 \to \Delta^0$ induces a map from objects $x \in \mathcal{C}$ to morphisms id_x (or 1_x). If $\mathcal{C}$ comes from the nerve of an ordinary category, this id_x just corresponds to the identity of the ordinary category.*

Definition B.5 *Let K be a simplicial set and $\mathcal{C}$ be an ∞-category. A functor $F\colon K \to \mathcal{C}$ is just a simplicial map. A natural transformation is a simplicial map $\Delta^1 \times K \to \mathcal{C}$. More generally, the space of functors $\mathrm{Fun}(K, \mathcal{C})$ is the mapping space $\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(K, \mathcal{C})$.*

Next, we define the homotopy category associated to an ∞-category:

Definition B.6 *Let $\mathcal{C}$ be an ∞-category and $x, y \in \mathcal{C}$. Two morphisms $f, g \colon x \to y$ are homotopic, denoted as $f \simeq g$, if there is a 2-simplex $\sigma \colon \Delta^2 \to \mathcal{C}$ such that $(\sigma \circ \delta_i)_{i=0,1,2}$ just correspond to (g, f, id_x) via Yoneda lemma, where the notation is taken from Example A.9.*

Proposition B.7 *The homotopy relation $\simeq$ of morphisms from x to y in $\mathcal{C}$ is an equivalence relation. The homotopy class of a morphism $f \colon x \to y$ is denoted by $[f]$.*

The homotopy category is defined by these homotopy classes:

Definition B.8 *Let $\mathcal{C}$ be an ∞-category. The homotopy category $\mathrm{h}\,\mathcal{C}$ is defined by:*

- *Objects: objects of $\mathcal{C}$;*
- *Morphisms: $\mathrm{Hom}_{\mathrm{h}\,\mathcal{C}}(x, y)$ is the homotopy classes of morphisms $x \to y$;*
- *Compositions: Let $f \colon x \to y$ and $g \colon y \to z$ be two morphisms. We form an inner horn $H \colon \Lambda_1^2 \to \mathcal{C}$ such that one face $H \circ \delta_0$ corresponds to g via Yoneda lemma, and another face $H \circ \delta_2$ corresponds to f. We extend H to $\tilde{H} \colon \Delta^2 \to \mathcal{C}$ and $\tilde{H} \circ \delta_1$ corresponds to a morphism $h \colon x \to z$. Then we impose $[h] = [g] \circ [f]$ (see Example A.9 for notations).*

This allows us to define the concept of a subcategory, which is not found in Groth's notes [4].

Definition B.9 *A full subcategory $\mathcal{D}$ of an ∞-category $\mathcal{C}$ is a simplicial subset of $\mathcal{C}$ spanned by a subset of vertices. In other words, a simplex of $\mathcal{C}$ lies in $\mathcal{D}$ if and only if all vertices of the simplex lies in $\mathcal{D}$.*

Definition B.10 ([9, Section 1.2.11]) *A subcategory $\mathcal{D}$ of an ∞-category $\mathcal{C}$ is a simplicial subset of $\mathcal{C}$ spanned by a subcategory of $\mathrm{h}\,\mathcal{C}$, which means that there is a subcategory $\mathcal{E} \subseteq \mathrm{h}\,\mathcal{C}$, such that a simplex of $\mathcal{C}$ lies in $\mathcal{D}$ if and only if all vertices of the simplex lies in $\mathcal{D}$, and for every edge f of the simplex, $[f] \in \mathcal{E}$.*

Definition B.11 *The core $\mathcal{C}^{\simeq}$ of an ∞-category $\mathcal{C}$, is the subcategory spanned by all objects and all equivalences. In other words, it is determined by the core of the homotopy category, $(\mathrm{h}\,\mathcal{C})^{\simeq}$. It is the maximal sub Kan complex of $\mathcal{C}$.*

We need the concept of mapping spaces between two objects in an ∞-category.

Definition B.12 *A simplicial category, or a simplicially enriched category, is a category enriched in the category of simplicial sets. To be more precise, a simplicial category $\mathcal{C}$ is given by the data of a set of objects $\mathrm{Ob}(\mathcal{C})$, a map $\mathrm{Map}_{\mathcal{C}} \colon \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}) \to \mathcal{S}\mathrm{et}_\Delta$ and a composition given by a collection of simplicial maps $(\circ \colon \mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{C}}(x, z))_{x,y,z \in \mathrm{Ob}(\mathcal{C})}$ satisfying*

- *The diagram*

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{C}}(z,w) \times \mathrm{Map}_{\mathcal{C}}(y,z) \times \mathrm{Map}_{\mathcal{C}}(x,y) & \xrightarrow{(\circ,\mathrm{id})} & \mathrm{Map}_{\mathcal{C}}(y,w) \times \mathrm{Map}_{\mathcal{C}}(x,y) \\
{\scriptstyle(\mathrm{id},\circ)}\big\downarrow & & \big\downarrow{\scriptstyle\circ} \\
\mathrm{Map}_{\mathcal{C}}(z,w) \times \mathrm{Map}_{\mathcal{C}}(x,z) & \xrightarrow{\circ} & \mathrm{Map}_{\mathcal{C}}(x,w)
\end{array}$$

  commutes for all $x, y, z, w \in \mathrm{Ob}(\mathcal{C})$*;*

- *For each* $x \in \mathrm{Ob}(\mathcal{C})$*, there exists a vertex* $\mathrm{id}_x \in \mathrm{Map}_{\mathcal{C}}(x,x)_0$ *(hence unique by a simple argument) such that* $\mathrm{id}_x \circ$ *and* $\circ\, \mathrm{id}_x$ *are identities.*

A functor $F\colon \mathcal{C} \to \mathcal{D}$ *between two simplicial categories is given by the data of a map* $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ *and a collection of simplical maps* $(\mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{D}}(Fx,Fy))_{x,y\in\mathrm{Ob}(\mathcal{C})}$ *each of which is compatible with the composition and preserves identities. An object* x *of* $\mathcal{C}$ *is usually denoted by* $x \in \mathcal{C}$ *instead of the clumsy notation* $x \in \mathrm{Ob}(\mathcal{C})$*. The category of simplicial categories, whose morphisms are functors is denoted by* $\mathrm{s}\mathcal{C}\mathrm{at}$*.*

Now we want to associate to an ∞-category a simplicial category such that simplicial sets $\mathrm{Map}(x,y)$ (or a fibrant replacement, because we usually call Kan complexes "spaces") just act as mapping spaces. More generally, we can do the same thing for general simplicial sets. To do this, we first associate a simplicial category to standard simplexes Δ^n, see [9, Definition 1.5.1.1] for a reference:

Definition B.13 *The simplicial category* $\mathfrak{C}[\Delta^n]$ *is defined by*

- $\mathrm{Ob}(\mathfrak{C}[\Delta^n]) = [n]$*;*
- *For* $0 \leq i,j \leq n$*,* $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j)$ *is empty if* $i > j$*, and* $\mathrm{N}(P_{i,j})$ *otherwise, where* $P_{i,j}$ *is the poset of subsets* $I \subseteq \{i, i+1, \ldots, j\}$ *such that* $i,j \in I$*, ordered by inclusion.*

$\mathfrak{C}$ *actually defines a functor* $\Delta \to \mathcal{S}\mathrm{et}_\Delta$*, where for each* $\alpha \in \Delta([m],[n])$*, we associate a natural functor* $\mathfrak{C}[\Delta^m] \to \mathfrak{C}[\Delta^n]$*.*

We extend the definition of $\mathcal{C}$ to any simplicial sets

Definition B.14 *Let* X *be a simplicial set. The associated simplicial category* $\mathfrak{C}[X]$*, is defined by the coend expression*

$$\mathfrak{C}[X] = \int^{[n]\in\Delta} X([n]) \times \mathfrak{C}[\Delta^n]$$

where the notation $X([n]) \times \mathfrak{C}[\Delta^n]$ *just denotes the "disjoint union" of* $X([n])$*'s copies of* $\mathfrak{C}[\Delta^n]$*.*

In practice, we need to compute mapping spaces. The preceding definition is not only a bit complicated, but also doesn't directly lead to Kan complexes. Instead, we define the space of right morphisms:

Definition B.15 ([9, Section 1.2.2]) *Let $\mathcal{C}$ be an ∞-category and $x, y \in \mathcal{C}$. The space of right morphisms from x to y is a simplicial set $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(x, y)$, defined as follows: the set of n-simplexes is defined to be the set of all $n+1$-simplexes of $\mathcal{C}$ such that the n-face spanned by first $n+1$ vertices is just the constant at x, and the last vertex is just y. Face and degeneracy maps are naturally defined.*

Proposition B.16 ([9, Proposition 1.2.2.3]) *Let $\mathcal{C}$ be an ∞-category and $x, y \in \mathcal{C}$. The space of right morphisms from x to y is a Kan complex, weak equivalent to $\mathrm{Map}_{\mathcal{C}}(x, y)$.*

Let's then define the join construction, the overcategory/undercategory and limits/colimits.

Definition B.17 ([4, Proposition 2.23]) *Let $\mathcal{C}$ be an ∞-category. The object $x \in \mathcal{C}$ is called final if for every simplicial sphere $\alpha\colon \partial\Delta^n \to \mathcal{C}$ such that $\alpha(n) = x$ can be extended into an entire n-complex $\Delta^n \to \mathcal{C}$. Similarly, $x \in \mathcal{C}$ is called initial if for every simplicial sphere $\alpha\colon \partial\Delta^n \to \mathcal{C}$ such that $\alpha(0) = x$ can be extended into an entire n-complex $\Delta^n \to \mathcal{C}$. A zero object of $\mathcal{C}$ is an object being both initial and final. We will say that $\mathcal{C}$ is pointed if it contains a zero object, usually denoted by $* \in \mathcal{C}$.*

Definition B.18 ([4, Definition 2.11]) *Let K and L be simplicial sets. The join $K \star L$ of K and L is the simplicial set defined by*

$$(K \star L)_n = K_n \cup L_n \cup \bigcup_{i+j+1=n} K_i \times L_j$$

the structure maps are defined naturally.

Geometrically, the $\|K \star L\|$ is the union of unit intervals l_{xy} connecting x, y where x run through points in $\|K\|$ and l runs through points in $\|L\|$. Especially, if K is a single point, then $K \star L$ is called the left cone on L; if L is a single point, then $K \star L$ is called the right cone on K.

Definition B.19 ([9, Prop 1.2.9.2, 1.2.9.3]) *Let $p\colon K \to \mathcal{C}$ be a map of simplicial sets and $\mathcal{C}$ be an ∞-category. The ∞-category of objects of $\mathcal{C}$ over p, $\mathcal{C}_{/p}$, is defined by the universal property:*

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(Y, \mathcal{C}_{/p}) \cong \mathrm{Hom}_{(\mathcal{S}\mathrm{et}_\Delta)_{K/}}(Y \star K, \mathcal{C})$$

functorial in $Y \in \mathcal{S}\mathrm{et}_\Delta$, where $K \to Y \star K$ is naturally defined. Similarly, the ∞-category of objects of $\mathcal{C}$ under p, $\mathcal{C}_{p/}$, is defined by replacing $Y \star K$ by $K \star Y$ in the preceding universal property.

Definition B.20 *Let $\mathcal{C}$ be an ∞-category, and $p\colon K \to \mathcal{C}$ a functor. A colimit for p is an initial object of $\mathcal{C}_{p/}$, and a limit for p is a final object of $\mathcal{C}_{/p}$.*

Definition B.21 *The product of a collection $(X_i)_{i\in I}$ of objects in an ∞-category $\mathcal{C}$ is defined to be a limit of the functor $I \to \mathcal{C}$ where I is viewed as a discrete simplicial set, usually denoted as $\prod_{i\in I} X_i$. If I is the finite set $\{1, 2, \ldots, n\}$, we also denote the product by $X_1 \times X_2 \times \cdots \times X_n$. Dually, the coproduct of $(X_i)_{i\in I}$ is a colimit of the functor in question, usually denoted as $\bigsqcup_{i\in I} X_i$, and if $I = \{1, 2, \ldots, n\}$, we denote the product by $X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$. If $\mathcal{C}$ is pointed, we usually denote the coproduct by $\bigvee_{i\in I} X_i$ and $X_1 \vee X_2 \vee \cdots \vee X_n$ respectively.*

In order to distinguish from the limits/colimits in ordinary categories, we usually call them homotopy limits/colimits.

Definition B.22 *An ∞-category $\mathcal{C}$ is called complete if it admits all limits, and cocomplete if it admits all colimits, finitely complete if it admits all finite limits (namely, the limits exist for any functor $K \to \mathcal{C}$ where K is a finite simplicial set), and finitely cocomplete if it admits all finite colimits.*

Example B.23 *Let $\mathcal{C}$ be an ordinary category. In $\mathrm{N}(\mathcal{C})$, the homotopy limits/colimits coincide with the usual limits/colimits in $\mathcal{C}$.*

We define the concept of a simplicial object in an ∞-category:

Definition B.24 *Let $\mathcal{C}$ be an ∞-category. A simplicial object in $\mathcal{C}$, is a functor $\mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}$.*

Definition B.25 *The geometric realization of a simplicial object $X\colon \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathcal{C}$, is simply the homotopy colimit of X.*

Definition B.26 ([8, Definition 1.1.1.4]) *Let $\mathcal{C}$ be a pointed ∞-category. A triangle in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ depicted as*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ * \arrow[r] & Z \end{tikzcd}$$

We will generally indicate a triangle by specifying only the pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ with the data of the diagram being implicitly assumed. We will say that a triangle in $\mathcal{C}$ is a fiber sequence if it is a pullback square, and a cofiber sequence if it is a pushout square.

Definition B.27 ([8, Definition 1.1.1.6]) *Let $\mathcal{C}$ be a pointed ∞-category containing a morphism $g\colon X \to Y$. A fiber of g is a fiber sequence $W \to X \xrightarrow{g} Y$, and a cofiber of g is a cofiber sequence $X \xrightarrow{g} Y \to Z$. By abuse of terminology, we also call W to be the fiber of g, and Z to be the cofiber of g, denoted by $W = \mathrm{fib}(g)$ and $Z = \mathrm{cofib}(g)$.*

Given these, we can define stable ∞-categories:

Definition B.28 ([8, Definition 1.1.1.9]) *An ∞-category $\mathcal{C}$ is stable if it satifies the following conditions:*

1. *It is pointed, i.e. there exists a zero object $* \in \mathcal{C}$;*
2. *Every morphism in $\mathcal{C}$ admits a fiber and a cofiber;*
3. *A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.*

Proposition B.29 ([8, Lemma 1.1.2.10]) *Let $\mathcal{C}$ be a stable ∞-category, then the homotopy category $\mathrm{h}\,\mathcal{C}$ is additive.*

Proposition B.30 ([8, Proposition 1.1.3.4]) *Let $\mathcal{C}$ be a pointed ∞-category. Then $\mathcal{C}$ is stable if and only if the following conditions are satisfied:*

1. *The ∞-category $\mathcal{C}$ is finitely complete and finitely cocomplete;*
2. *A square $\Delta^1 \times \Delta^1 \to \mathcal{C}$ is a pushout if and only if it is a pullback.*

Bibliography

[1] C. Barwick. "On the Q construction for exact quasicategories." In: *ArXiv e-prints* (Jan. 2013). arXiv: 1301.4725 [math.KT].

[2] Clark Barwick. "On the algebraic K-theory of higher categories." In: *Journal of Topology* 9.1 (2016), pp. 245–347. DOI: 10.1112/jtopol/jtv042. URL: http://dx.doi.org/10.1112/jtopol/jtv042.

[3] D. Gepner, M. Groth, and T. Nikolaus. "Universality of multiplicative infinite loop space machines." In: *ArXiv e-prints* (May 2013). arXiv: 1305.4550 [math.AT].

[4] M. Groth. "A short course on ∞-categories." In: *ArXiv e-prints* (July 2010). arXiv: 1007.2925 [math.AT].

[5] André Joyal and Myles Tierney. "Quasi-categories vs. Segal spaces." In: *Categories in algebra, geometry and mathematical physics*. Ed. by Alexei Davydov et al. Contemporary Mathematics 431. 2007, pp. 277–326. arXiv: math/0607820 [math.AT].

[6] Bernhard Keller. "Chain complexes and stable categories." In: *manuscripta mathematica* 67.1 (1990), pp. 379–417. ISSN: 1432-1785. DOI: 10.1007/BF02568439. URL: https://doi.org/10.1007/BF02568439.

[7] Jacob Lurie. *Algebraic K-Theory and Manifold Topology*. URL: http://www.math.harvard.edu/~lurie/281.html.

[8] Jacob Lurie. "Higher Algebra." Mar. 2017.

[9] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009.

[10] Paul G. Goerss, John Jardine. *Simplicial Homotopy Theory*. First. Birkhäuser Basel, 2009.

[11] Daniel Quillen. "Higher algebraic K-theory: I." In: *Higher K-Theories: Proceedings of the Conference held at the Seattle Research Center of the Battelle Memorial Institute, from August 28 to September 8, 1972*. Ed. by H. Bass. Berlin, Heidelberg: Springer Berlin Heidelberg, 1973, pp. 85–147. ISBN: 978-3-540-37767-2. DOI: 10.1007/BFb0067053. URL: https://doi.org/10.1007/BFb0067053.

[12] Daniel Quillen. "The Spectrum of an Equivariant Cohomology Ring: I." In: *Annals of Mathematics* 94.3 (1971), pp. 549–572. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970770.

[13] Daniel Quillen. "The Spectrum of an Equivariant Cohomology Ring: II." In: *Annals of Mathematics* 94.3 (1971), pp. 573–602. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970771.

[14] Charles Rezk. "A model for the homotopy theory of homotopy theory." In: *Trans. Amer. Math. Soc.* 353.3 (2001), pp. 973–1007. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-00-02653-2. URL: http://dx.doi.org/10.1090/S0002-9947-00-02653-2.

[15] Graeme Segal. "Categories and cohomology theories." In: *Topology* 13.3 (1974), pp. 293 –312. ISSN: 0040-9383. DOI: http://dx.doi.org/10.1016/0040-9383(74)90022-6. URL: http://www.sciencedirect.com/science/article/pii/0040938374900226.

[16] V. Srinivas. *Algebraic K-Theory*. Second. Birkhäuser Basel, 1996.

[17] Friedhelm Waldhausen. "Algebraic K-theory of spaces." In: *Algebraic and geometric topology: proceedings of a conference held at Rutgers Univ., New Brunswick, USA, July 6 - 13, 1983*. Ed. by Andrew Ranicki, Norman Levitt, and Frank Quinn. Springer, 1985, pp. 318–419.