EQUIVARIANT ASPECTS OF DE-COMPLETING CYCLIC HOMOLOGY

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ABSTRACT. Derived de Rham cohomology turns out to be important in p-adic geometry, following Bhatt's discovery [Bha12] of conjugate filtration in char p, de-Hodge-completing results in [Bei12]. In [Kal18], Kaledin introduced an analogous de-completion of the periodic cyclic homology, called the polynomial periodic cyclic homology, equipped with a conjugate filtration in char p, and expected to be related to derived de Rham cohomology.

In this article, using genuine equivariant homotopy structure on Hochschild homology as in [ABG+18, BHM22], we give an equivariant description of Kaledin's polynomial periodic cyclic homology. This leads to Morita invariance without any Noetherianness assumption as in [Kal18], and the comparison to derived de Rham cohomology becomes transparent. Moreover, this description adapts directly to "topological" analogues, which gives rise to a de-Nygaardcompletion of the topological periodic cyclic homology over a perfectoid base. We compare it to topological Hochschild homology over \mathbb{F}_p , and produce a conjugate filtration in char p from our description.

1. INTRODUCTION

Grothendieck's algebraic de Rham cohomology, introduced in [Gro66], turns out to be an important tool to study the cohomology of smooth schemes. However, it does not behave well beyond the smooth case. Illusie, following ideas of Quillen, introduced *derived de Rham cohomology*, along with its Hodge-completion, in [Ill72, Ch. VIII].

Hodge-completion makes derived de Rham cohomology easier to control, since the associated graded pieces are given by shifts of (derived) exterior powers of the cotangent complex. In particular, it coincides with algebraic de Rham cohomology for smooth schemes.

On the other hand, (non-Hodge-completed) derived de Rham cohomology was intractable until Bhatt's discovery in [Bha12] of *conjugate filtration* on it in char p, whose associated graded pieces are equivalent to shifts of Frobenius twists of algebraic differential forms. He also observed the triviality of derived de Rham cohomology after rationalization. Using this new tool, he identified derived de Rham cohomology with crystalline cohomology for lci maps between \mathbb{Z}/p^r -schemes. Later on, this non-Hodge-complete version becomes useful in p-adic geometry. For example, Fontaine's period rings $A_{\rm cris}$ and $C_{\rm st}$ are equipped with non-complete Hodge-filtration, and Bhatt applied this non-Hodge-complete version to prove some Beilinson's conjectures in [Bei12].

Periodic cyclic homology is a noncommutative counterpart of Hodge-completed derived de Rham cohomology, defined for general DG-categories. For morphisms of Q-schemes, the relation is particularly simple: periodic cyclic homology is a product

Note: This article has been written using GNU $T_{E}X_{MACS}$ [Hoe20].

of shifts of derived de Rham cohomology, as recently shown by Konrad BALS in full generality in [Bal24]. This relation was firstly discovered by Loday–Quillen for smooth morphisms of Q-schemes, cf. [Lod98, §5.1.12], and studied in [TV11]. For schemes beyond char 0, it was studied in [Maj96]. Breakthroughs were made in [BMS19, Ant19], which proved that, there is a complete filtration on periodic cyclic homology, whose associated graded pieces are shifts of Hodge-completed derived de Rham cohomology.

In view of usefulness of non-Hodge-completed derived de Rham cohomology in *p*-adic geometry, it is natural to ask whether there is a "non-Hodge-completion" of periodic cyclic homology for DG-categories, which carries a filtration with associated graded pieces being shifts of non-Hodge-completed derived de Rham cohomology? In [Kal18], following Kontsevich's suggestion [Kon08, 2.32], Kaledin defined *polynomial periodic cyclic homology*, equipped with a conjugate filtration in char p (when p = 2, it is later constructed in [Kal17]), whose associated graded pieces are equivalent to shifts of Frobenius twisted Hochschild homology. Using this, he showed that the conjugate-completion of polynomial periodic cyclic homology, called *co-periodic cyclic homology*, is a derived Morita invariant when the base is Noetherian. Moreover, he expected that polynomial periodic cyclic homology is closely related to derived de Rham cohomology.

As explained in [Kal18], Kaledin's defines polynomial periodic cyclic homology and deals with it by explicit manipulations of chain complexes, which makes the arguments technical and difficult, and the homotopy-theoretic functoriality of this construction becomes opaque. The main goal of this article is to give a "more invariant treatment" as he wished in the introduction, which overcomes these difficulties.

The key to our description is the genuine equivariant homotopy structure on the usual \mathbb{Z} -linear Hochschild homology. More precisely, let \mathcal{C} be a DG-category. Then the Hochschild homology of \mathcal{C} , being a Borel \mathbb{T} -equivariant \mathbb{Z} -module spectrum, has the formula

$$\operatorname{HH}(\mathcal{C}/\mathbb{Z}) = \operatorname{THH}(\mathcal{C}) \otimes_{\operatorname{THH}(\mathbb{Z})}^{\mathbb{L}} \mathbb{Z}.$$

This admits an obvious cyclonic (à la [BG16]) structure: the \mathbb{T} -equivariant ring \mathbb{Z} is the underlying object of the constant Tambara functor \mathbb{Z} , and the universal property of THH in [ABG+18] gives rise to a map THH(\mathbb{Z}) $\rightarrow \mathbb{Z}$ of \mathbb{T} - \mathbb{E}_{∞} -rings. This gives rise to an enhancement

$$\operatorname{HH}(\mathcal{C}/\underline{\mathbb{Z}}) = \operatorname{THH}(\mathcal{C}) \otimes_{\operatorname{THH}(\mathbb{Z})}^{\mathbb{L}} \underline{\mathbb{Z}}$$

as a $\underline{\mathbb{Z}}$ -module in cyclonic spectra. This was generalized to a genuine version of factorization homology in [BHM22]. However, up to our knowledge, such a genuine equivariant homotopy structure on \mathbb{Z} -linear Hochschild homology does not seem to be studied in the literature. By definition, such an enhancement is a derived Morita invariant.

Remark 1.1. This genuine equivariant homotopy structure has other applications. In a companion paper [Mao24], we use a "thickening" of it to define prismatic Hochschild homology. In our forthcoming paper [Mao], we use a similar genuine equivariant structure to streamline Kaledin's Hochschild–Witt homology, a noncommutative counterpart of de Rham–Witt complex. It turns out that such a structure contains enough information to recover polynomial periodic cyclic homology. Recall that, in terms of explicit chain complexes, the usual Tate construction involves a product totalization in one direction, and Kaledin's polynomial periodic cyclic homology is taking the direct sum totalization instead, so that it has good colimit-preserving properties. Inspired by this, we introduce the following definitions:

Definition 1.2. (Definitions 2.1 and 3.3) The ($\underline{\mathbb{Z}}$ -)de-completed \mathbb{T} -Tate construction $(-)^{\theta_{\underline{\mathbb{Z}}}}\mathbb{T}$ is the filtered-colimit-preserving approximation of the composite functor

$$\operatorname{Mod}_{\mathbb{Z}}(\operatorname{Sp}^{g^{<}}\mathbb{T}) \longrightarrow \operatorname{Sp}^{B^{\mathbb{T}}} \xrightarrow{(-)^{t^{\mathbb{T}}}} \operatorname{Sp}^{t^{\mathbb{T}}}$$

The polynomial periodic cyclic homology $\operatorname{HP}^{\operatorname{poly}}(\mathcal{C}/\mathbb{Z})$ of a DG-category \mathcal{C} is defined to be $\operatorname{HH}(\mathcal{C}/\underline{\mathbb{Z}})^{\theta_{\underline{\mathbb{Z}}}}\mathbb{T}$, applying the de-completed \mathbb{T} -Tate construction to Hochschild homology $\operatorname{HH}(\mathcal{C}/\underline{\mathbb{Z}})$.

The same construction works for any t-bounded animated ring as base in place of \mathbb{Z} , as done in the main text.

Remark 1.3. The de-completed Tate construction depends on the choice of base. However, in some cases, it does not quite depend on that. We will prove relevant results in Section 4.

From this description, it is immediate that polynomial periodic cyclic homology is rationally zero (Remark 2.9), since the functor $\operatorname{Mod}_{\mathbb{Z}}(\operatorname{Sp}^{g^{<}\mathbb{T}}) \to \operatorname{Sp}^{B\mathbb{T}}$ becomes an equivalence after rationalization. With slightly more efforts, we show that

Proposition 1.4. (Proposition 4.4) Let C be a smooth and bounded DG-category^{1,1}. Then the assembly map $\operatorname{HP^{poly}}(C/\mathbb{Z}) \to \operatorname{HP}(C/\mathbb{Z})$ is an equivalence after profinite completion.

Note that, for every quasicompact quasiseparated scheme X, its derived category D(X) is bounded, by [BvdB03, Cor 3.1.8]. When X is in addition smooth, then its derived category D(X) is also smooth. We refer to [Orl16] for general discussions. This proposition, along with colimit-preservation of de-completed Tate construction, implies that, on animated rings, the polynomial periodic cyclic homology is left Kan extended from polynomial rings, and thus it coincides with various adhoc constructions in the literature^{1.2}, such as in [BMS19, AMMN22]. Consequently, we address Kaledin's expectation in the following.

Proposition 1.5. (Construction 4.7) Let R be a commutative ring. Then there exists a functorial filtration $\operatorname{Fil}^*_{\mathrm{HKR}}$ on the profinite completion $\operatorname{HP}^{\mathrm{poly}}(R/\mathbb{Z})^{\wedge}$ of polynomial periodic cyclic homology with associated graded pieces equivalent to shifts of derived de Rham cohomology $\operatorname{dR}_{R/\mathbb{Z}}$ after profinite completion.

Our description also suggests a "topological" analogue.

Definition 1.6. (Definition 4.8) Let S be a perfectoid ring, and C a DG-category over S. Then topological polynomial periodic cyclic homology $\mathrm{TP}^{\mathrm{poly}/S}(\mathcal{C})$ is defined to be $\mathrm{THH}(\mathcal{C})^{\theta_{\mathrm{THH}(S)}\mathbb{T}}$, defined by applying $\mathrm{THH}(S)$ -de-completed \mathbb{T} -Tate construction to topological Hochschild homology $\mathrm{THH}(\mathcal{C})$.

^{1.1.} Smooth DG-categories are necessarily modules categories over smooth \mathbb{E}_1 -rings, by [Lur18, Prop 11.3.2.4]. A DG-category is *bounded* if, for every pair (x, y) of compact objects, the mapping \mathbb{Z} -module spectrum $\operatorname{Hom}(x, y) \in D(\mathbb{Z})$ has bounded Tor-amplitude (or equivalently, *t*-bounded, since \mathbb{Z} is of finite flat dimension).

^{1.2.} Also compare with [Man24, §1], and Devalapurkar–Hahn–Raksit–Yuan as mentioned in [Man24, Rem 1.6].

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Note that this is an arena where explicit chain complex manipulations cannot arrive. Previous results for polynomial periodic cyclic homology adapts to its topological analogue as well:

Proposition 1.7. (Corollary 4.11) Let S be a perfectoid ring, and R a pcompletely smooth and bounded DG-category over S. Then the assembly map $\mathrm{TP}^{\mathrm{poly}/S}(R) \to \mathrm{TP}(R)$ is an equivalence after $(p, \ker(\theta))$ -completion, where θ : $A_{\mathrm{inf}}(S) \twoheadrightarrow S$ is Fontaine's map.

Proposition 1.8. (Construction 4.12) Let S be a perfectoid ring, and R a commutative S-algebra. Then there exists a functorial filtration Fil_M^* on the p-completed topological polynomial periodic cyclic homology $\operatorname{TP}^{\operatorname{poly}/S}(R)_p^{\wedge}$ with associated graded pieces equivalent to shifts of Frobenius twisted prismatic cohomology $\varphi_A^* \triangle_{R/A}$ after p-completion, where $A := A_{\inf}(S)$.

It seems slightly surprising that cyclotomic THH (with THH(S)-module structure) is already enough to de-Nygaard-complete topological periodic cyclic homology, but this phenomenon is demystified by Efimov's rigidity of localizing motives (Remark 4.16).

As for noncommutative geometry on its own, we give two proof for the following comparison, due to Kaledin in [Kal20, Cor 11.15], but our proof is much simpler.

Proposition 1.9. (Corollary 5.10) Let C be a DG-category over \mathbb{F}_p . Then the polynomial periodic cyclic homology $\operatorname{HPpoly}(\mathcal{C}/\mathbb{F}_p)$ is equivalent to $\operatorname{THH}(\mathcal{C})[\sigma^{-1}]$ as $\mathbb{Z}^{t\mathbb{T}}$ -module spectra.

We also produce a conjugate filtration on polynomial periodic cyclic homology in char p in Section 6, and prove that

Proposition 1.10. (Corollary 6.21) Let k be a commutative \mathbb{F}_p -algebra, and C a DG-category over k. Then the conjugate filtration on $\operatorname{HP}^{\operatorname{poly}}(\mathcal{C}/k)$ is complete in the following two cases:

- 1. C = D(R) for some (-1)-connective \mathbb{E}_1 -k-algebra R (which includes all associative k-algebras R);
- 2. C = D(X) for a quasicompact quasiseparated k-scheme X.

Remark 1.11. In comparison Efimov's refined negative cyclic homology and its continuation in Scholze's ongoing work^{1.3} on refined TC⁻, as mentioned above, in Remark 4.16, we explain that topological Hochschild homology as a cyclotomic spectrum already sees "all" *p*-adic formal information. However, their versions capture rigid analytic information. For example, for smooth \mathbb{F}_p -schemes, their versions see rigid cohomology. It might be worth understanding whether equivariant homotopy theory could say something for their versions as well.

Notation 1.12. Let G be a finite group. We denote by Sp^{gG} the symmetric monoidal ∞ -category of genuine G-spectra, by $\operatorname{Mack}_{G}^{\operatorname{coh}}(k)$ the abelian category of k-linear cohomological G-Mackey functors (we omit k when $k = \mathbb{Z}$), and by $\operatorname{Sp}^{g^{<}\mathbb{T}}$ (resp. $\operatorname{Sp}^{g_{p}\mathbb{T}}$) the symmetric monoidal ∞ -category of cyclonic (resp. p-cyclonic) spectra as in [BG16].

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^{1.3.} An abstract, along with recordings, of the talk can be found at https://www.mpimbonn.mpg.de/node/13359.

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2. A de-completion of Tate construction

Let k be a commutative ring, and G a finite group or \mathbb{T} . Recall that the G-Tate construction $(-)^{tG}: D(k)^{BG} \to D(k)$ does not preserve filtered colimits. In this section, we try to "de-complete" it when the input is further equipped with a genuine equivariant structure. When $G = C_p$, we will show that it can be expressed in terms of the geometric fixed points. We will also establish a de-completed version of the Tate orbit lemma.

Definition 2.1. Let G be a finite group (resp. \mathbb{T}), and A an \mathbb{E}_1 -algebra in the symmetric monoidal ∞ -category Sp^{gG} (resp. $\operatorname{Sp}^{g^{<}\mathbb{T}}$) of G-spectra (resp. cyclonic spectra). Then

- The (A-)de-completed homotopy G-fixed points $(-)^{\eta_A G}$: RMod_A(Sp^{gG}) \rightarrow Sp (resp. RMod_A(Sp^{g[<]T}) \rightarrow Sp) is the filtered-colimit-preserving approximation of the homotopy G-fixed points $(-)^{hG}$: RMod_A(Sp^{gG}) \rightarrow Sp (resp. RMod_A(Sp^{g[<]T}) \rightarrow Sp), equipped with an assembly map $(-)^{\eta_A G} \rightarrow (-)^{hG}$. We omit "A-" when the context is clear.
- The (A-)de-completed G-Tate construction $(-)^{\theta_A G}$: $\operatorname{RMod}_A(\operatorname{Sp}^{gG}) \to \operatorname{Sp}(\operatorname{resp.} \operatorname{RMod}_A(\operatorname{Sp}^{g^{<}\mathbb{T}}) \to \operatorname{Sp})$ is the filtered-colimit-preserving approximation of the G-Tate construction $(-)^{tG}$: $\operatorname{RMod}_A(\operatorname{Sp}^{gG}) \to \operatorname{Sp}(\operatorname{resp.} \operatorname{RMod}_A(\operatorname{Sp}^{g^{<}\mathbb{T}}) \to \operatorname{Sp})$, equipped with an assembly map $(-)^{\theta_A G} \to (-)^{tG}$ which canonically fits into a commutative diagram

$$\begin{array}{cccc} (-)^{\eta_A G} & \longrightarrow & (-)^{hG} \\ \downarrow & & \downarrow \\ (-)^{\theta_A G} & \longrightarrow & (-)^{tG} \end{array}$$

$$(2.1)$$

Remark 2.2. Let G be a finite cyclic group (resp. \mathbb{T}), and A a commutative algebra in $\mathcal{E} := \operatorname{Sp}^{gG}$ (resp. $\mathcal{E} := \operatorname{Sp}^{g \leq \mathbb{T}}$). Then the diagram (2.1) is Cartesian in the ∞ -category Fun^{Ex}(\mathcal{E} , Sp): the fiber of the canonical natural transformation $(-)^{hG} \to (-)^{tG}$ is $(-)_{hG} \in \operatorname{Fun}^{\operatorname{Ex}}(\mathcal{E}, \operatorname{Sp})$, which preserves filtered colimits as well, and consequently, the diagram (2.1) induces an equivalence on fibers of vertical arrows.

Remark 2.3. Let G be a finite group (resp. T), and $A \to B$ a map of \mathbb{E}_1 -algebras in the symmetric monoidal ∞ -category $\mathcal{E} := \operatorname{Sp}^{gG}$ (resp. $\mathcal{E} := \operatorname{Sp}^{g^{<}T}$) of G-spectra (resp. cyclonic spectra). Then the assembly maps on compact objects of $\operatorname{RMod}_B(\mathcal{E})$ induces "relative" assembly maps $(-)^{\eta_A G} \to (-)^{\eta_B G}$ and $(-)^{\theta_A G} \to (-)^{\theta_B G}$, which fits into a commutative diagram

$$\begin{array}{ccc} (-)^{\eta_A G} & \longrightarrow & (-)^{\eta_B G} \\ \downarrow & & \downarrow \\ (-)^{\theta_A G} & \longrightarrow & (-)^{\theta_B G} \end{array}$$

which is Cartesian by Remark 2.2.

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Remark 2.4. Let G be a finite group (resp. T), and A an \mathbb{E}_1 -algebra in the symmetric monoidal ∞ -category $\mathcal{E} := \operatorname{Sp}^{gG}$ (resp. $\mathcal{E} := \operatorname{Sp}^{g^{\leq T}}$) of G-spectra (resp. cyclonic spectra). Let $A \to A^h$ denote the Borel completion of A. Then it follows immediately from the definitions that the de-completed homotopy G-fixed points $(-)^{\eta_A G}$ factors as

$$\operatorname{RMod}_{A}(\mathcal{E}) \xrightarrow{(-)\otimes_{A}^{\mathbb{L}}A^{h}} \operatorname{RMod}_{A^{h}}(\mathcal{E}) \xrightarrow{\eta_{A^{h}}G} \operatorname{Sp}$$

and similarly for the de-completed G-Tate construction. Roughly speaking, it does not hurt to replace all genuine equivariant bases by their Borel completions. However, sometimes it seems to be convenient to consider genuine equivariant bases.

Remark 2.5. Let G be a finite group, and A an \mathbb{E}_{∞} -algebra in Sp^{gG} . Then the lax symmetric monoidal structure on the homotopy fixed points $(-)^{hG}$ (resp. the Tate construction $(-)^{tG}$) gives rise^{2.1} to a lax symmetric monoidal structure on the decompleted homotopy fixed points $(-)^{\eta_A G}$ (resp. the de-completed Tate construction $(-)^{\theta_A G}$). The assembly maps are equipped with a lax symmetric monoidal structure as well. In particular, the objects $M^{\eta_A G}$ (resp. $M^{\theta_A G}$) carries a canonical A^{hG} -(resp. A^{tG} -)module structure, which is functorial in $M \in \mathrm{Mod}_A(\mathrm{Sp}^{gG})$.

Remark 2.6. In desirable situations, the A-de-completed homotopy G-fixed points (resp. G-Tate construction) does not quite depend on A. We discuss some independences of this form in Section 4.

Remark 2.7. Let G be a finite group, and A an \mathbb{E}_{∞} -algebra in Sp^{gG} . Recall that the symmetric monoidal ∞ -category Sp^{gG} is rigid, thus the forgetful functor $\mathrm{Mod}_A(\mathrm{Sp}^{gG}) \to \mathrm{Mod}_A(\mathrm{Sp}^{BG})$ factors through the rigidification $(\mathrm{Mod}_A(\mathrm{Sp}^{BG}))^{\mathrm{rig}}$ of the target, and the A-de-completed G-Tate construction $(-)^{\theta_A G}$ coincides with the composite

$$\operatorname{Mod}_A(\operatorname{Sp}^{gG}) \longrightarrow (\operatorname{Mod}_A(\operatorname{Sp}^{BG}))^{\operatorname{rig}} \xrightarrow{(-)^{tG,\operatorname{rig}}} \operatorname{Sp},$$

which can be checked by restricting to compact objects of $Mod_A(Sp^{gG})$. The same holds for de-completed homotopy *G*-fixed points.

Remark 2.8. Let A be an \mathbb{E}_{∞} -algebra in $D(\mathbb{Z}) \otimes \operatorname{Sp}^{g^{\leq} \mathbb{T}}$. Recall that, for every A-module M in $\operatorname{Sp}^{g^{\leq} \mathbb{T}}$ and every positive integer $n \in \mathbb{N}_{>0}$, the canonical maps

$$M^{t\mathbb{T}}/n \longleftarrow M^{t\mathbb{T}} \otimes_{A^{t\mathbb{T}}} A^{tC_n} \longrightarrow M^{tC_r}$$

are equivalences, [NS18, Lem IV.4.12]. It follows that, the canonical maps

$$M^{\theta_A \, \mathbb{T}} \, / \, n \longleftarrow M^{\theta_A \, \mathbb{T}} \otimes_{A^{t \mathbb{T}}} A^{t C_n} \longrightarrow M^{\theta_A C_n}$$

are equivalences as well. Consequently, up to profinite completion, the A-de-completed \mathbb{T} -Tate construction $M^{\theta_A \mathbb{T}}$ can be recovered from de-completed C_n -Tate constructions, where *n* runs through all positive integers. This gives rise to a lax symmetric monoidal structure on the profinitely completed de-completed \mathbb{T} -Tate constructure $(M^{\theta_A \mathbb{T}})^{\wedge}$.

Remark 2.9. Let k be a commutative ring. Then the constant Green functor \underline{k} can be viewed as an object of $D(\mathbb{Z}) \otimes \operatorname{Sp}^{g^{\leq T}}$. Then for every $n \in \mathbb{N}_{>0}$, we have

$$(\underline{k} \otimes \Sigma^{\infty}_{\mathbb{T}} [\mathbb{T}/C_n]_+)^{h\mathbb{T}} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q} \simeq 0 \simeq (\underline{k} \otimes \Sigma^{\infty}_{\mathbb{T}} [\mathbb{T}/C_n]_+)^{t\mathbb{T}} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}.$$

^{2.1.} Here the same argument does not work for $G = \mathbb{T}$.

Consequently, the rationalized de-completed \mathbb{T} -Tate construction $(-)^{\theta_{\underline{k}}\mathbb{T}}$ vanishes. Combining with Remark 2.8, we see that the de-completed \mathbb{T} -Tate construction $(-)^{\theta_{\underline{k}}\mathbb{T}}$ can be completely recovered from de-completed C_n -Tate constructions, where n runs through all positive integers.

The Tate orbit lemma admits a de-completion. We first observe that, since C_p is a simple group^{2.2}, the de-complete C_p -Tate construction has a fairly simple formula:

Lemma 2.10. Let A be an \mathbb{E}_1 -algebra in the symmetric monoidal category Sp^{gC_p} of C_p -spectra. Then the A-de-completed C_p -Tate construction $(-)^{\theta_A C_p}$ coincides with the composite functor

$$\operatorname{RMod}_A(\operatorname{Sp}^{gC_p}) \xrightarrow{(-)^{\Phi C_p}} D(A^{\Phi C_p}) \longrightarrow D(A^{tC_p})$$

where the second functor is the base change along the map $A^{\Phi C_p} \to A^{tC_p}$ of \mathbb{E}_1 -rings. Moreover, when A is an \mathbb{E}_{∞} -algebra in Sp^{gC_p} , this identification is (lax) symmetric monoidal.

Proof. It suffices to restrict to compact objects of $\operatorname{RMod}_A(\operatorname{Sp}^{gC_p})$. For compact objects, there are many ways to see this. For example, the functor $(-)^{tC_p}$ vanishes on induced C_p -spectra, thus it canonically factors through $D(A^{\Phi C_p})$ in Pr^L (here we use the fact that C_p is a simple group), cf. [AMR21, §5], and then to see that the result functor $D(A^{\Phi C_p}) \to D(A^{tC_p})$ in Pr^L coincides with the base change, it suffices to check on the generator, which is straightforward. The symmetric monoidal structure follows from a similar argument.

Corollary 2.11. Let A be an \mathbb{E}_1 -algebra in the symmetric monoidal category Sp^{gC_p} of C_p -spectra. Then the A-de-completed C_p -Tate construction $(-)^{\theta_A C_p}$: $\operatorname{RMod}_A(\operatorname{Sp}^{gC_p}) \to D(A^{tC_p})$ is strongly continuous^{2.3}.

Corollary 2.12. Let A be an \mathbb{E}_{∞} -algebra in the symmetric monoidal category Sp^{gC_p} of C_p -spectra. Then the A-de-completed C_p -Tate construction $(-)^{\theta_A C_p}$ is symmetric monoidal.

Lemma 2.13. Let A be a bounded below \mathbb{E}_1 -algebra in cyclonic spectra. Then the p-completed A-de-completed \mathbb{T} -Tate construction^{2.4}

$$((-)^{\theta_{A}\mathbb{T}})_{p}^{\wedge} : \operatorname{RMod}_{A}(\operatorname{Sp}^{g^{<}\mathbb{T}}) \longrightarrow D(A^{t\mathbb{T}})_{p}^{\wedge},$$

factors $as^{2.5}$

$$\operatorname{RMod}_A(\operatorname{Sp}^{g^{<}\mathbb{T}}) \xrightarrow{(-)^{\theta_A C_p}} \operatorname{RMod}_{A^{tC_p}}(\operatorname{Sp}^{g_p(\mathbb{T}/C_p)}) \xrightarrow{(-)^{\eta_A tC_p(\mathbb{T}/C_p)}} D(A^{t\mathbb{T}})_p^{\wedge}$$

in $\operatorname{Pr}_{\operatorname{St}}^{L}$.

^{2.2.} In this article, we only consider cyclic group actions, but the argument works for any simple group.

^{2.3.} A functor $F: \mathcal{C} \to \mathcal{D}$ in $\operatorname{Pr}^{R}_{\operatorname{St}}$ is *strongly continuous* if its right adjoint $F^{R}: \mathcal{D} \to \mathcal{C}$ preserves filtered colimits. When \mathcal{C} is compactly generated, it is equivalent to F preserving compact objects.

^{2.4.} Note that the forgetful functor $\operatorname{Sp}^{g^{<}\mathbb{T}} \to \operatorname{Sp}^{g_{p}\mathbb{T}}$ induces an equivalence on *p*-complete objects, thus we could work *p*-typically throughout.

^{2.5.} Thanks to Remark 2.4, it does not matter what cyclonic structure on A^{tC_p} that we put.

Proof. Since every functor preserves filtered colimits, it suffices to check on compact objects in $\operatorname{RMod}_A(\operatorname{Sp}^{g^{<}\mathbb{T}})$, and this follows from Corollary 2.11 and the Tate orbit lemma (since A is bounded below, so is $A \otimes \Sigma_{\mathbb{T}}^{\infty} [\mathbb{T}/C_m]_+$ for every $m \in \mathbb{N}_{>0}$). \Box

The same argument works for finite cyclic groups (for which we can further keep track of the lax symmetric monoidal structure on the de-completed Tate construction).

Lemma 2.14. Let $r \in \mathbb{N}_{>0}$, and A a bounded below \mathbb{E}_1 -(resp. \mathbb{E}_{∞} -)algebra in C_{p^r} -spectra. Then the A-de-completed C_{p^r} -Tate construction

$$(-)^{\theta_A \mathbb{T}} : \operatorname{Mod}_A(\operatorname{Sp}^{C_{p^r}}) \longrightarrow D(A^{tC_{p^r}}),$$

as a presentable (resp. lax symmetric monoidal) functor, factors as

$$\operatorname{Mod}_{A}(\operatorname{Sp}^{gC_{p^{r}}}) \xrightarrow{(-)^{\theta_{A}C_{p}}} \operatorname{Mod}_{A^{tC_{p}}}(\operatorname{Sp}^{g_{p}(C_{p^{r}}/C_{p})}) \xrightarrow{(-)^{\eta_{A}tC_{p}(C_{p^{r}}/C_{p})}} D(A^{tC_{p^{r}}}).$$

Remark 2.15. Let k be a commutative ring. By Remarks 2.8 and 2.9 and Lemma 2.14, we can also keep track of the lax symmetric monoidal structure on the p-completed de-completed \mathbb{T} -Tate construction $((-)^{\theta_{\underline{k}}\mathbb{T}})_p^{\wedge}$, showing that, as an exact lax symmetric monoidal functor, it factors through the composite exact symmetric monoidal functor

$$\operatorname{Mod}_{\underline{k}}(\operatorname{Sp}^{g^{<}}\mathbb{T}) \xrightarrow{(-)^{\Phi C_{p}}} \operatorname{Mod}_{\underline{k}^{\Phi C_{p}}}(\operatorname{Sp}^{g^{<}}(\mathbb{T}/C_{p})) \longrightarrow \operatorname{Mod}_{k^{t}C_{p}}(\operatorname{Sp}^{g^{<}}(\mathbb{T}/C_{p}))$$

where the remaining functor $\operatorname{Mod}_{k^{tC_p}}(\operatorname{Sp}^{g^{\leq}(\mathbb{T}/C_p)}) \to D(A)$ is the limit of de-completed homotopy C_{p^r} -fixed points along $r \in \mathbb{N}$ (which at least a priori does not necessarily preserve filtered colimits).

3. Polynomial periodic and negative cyclic homology

We briefly review Kaledin's *polynomial periodic cyclic homology* of cyclic objects, and then describe it in terms of the de-completed \mathbb{T} -Tate construction, informally explaining why it coincides with Kaledin's original construction.

Let k be a commutative ring, and $X_{\bullet}: \Lambda^{\mathrm{op}} \to \mathrm{Mod}_k$ a cyclic objects in k-modules. Recall that, for every $[n]_{\Lambda} \in \Lambda^{\mathrm{op}}$, the k-module $X_n := X_{\bullet}([n]_{\Lambda})$ carries an k-linear C_n action, which gives rise to a 2-periodic complex

$$\cdots \xleftarrow{1-\sigma_n} X_n \xleftarrow{N_n} X_n \xleftarrow{1-\sigma_n} X_n \xleftarrow{N_n} \cdots$$
weight
$$-1 \qquad 0 \qquad 1$$

of k-modules, where $\sigma_n: X_n \to X_n$ is the generator of C_n , and $N_n:=1+\sigma_n+\cdots+\sigma_n^{n-1}$ is the C_n -norm. This complex represents the shifted Tate construction $X_n^{tC_n}[-1]$. These complexes compile into a double complex

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where we surpress the subscripts of N and σ , and vertical differentials are appropriately 2-periodically given. Then

- The periodic cyclic homology $\operatorname{HP}(X_{\bullet}/k)$ of the cyclic object X_{\bullet} , is the object in the derived category D(k) of k-modules represented by the product totalization of this double complex.
- The polynomial periodic cyclic homology $\operatorname{HP^{poly}}(X_{\bullet}/k)$ of the cyclic object X_{\bullet} is the object in the derived category D(k) of k-modules represented by the direct sum totalization of this double complex.

We now give an alternative, more conceptual description of this double complex and the polynomial periodic cyclic homology. Recall that, the geometric realization $|X_{\bullet}|_{\Lambda} \in D(k)^{B^{\mathrm{T}}}$ can be rewritten as a geometric realization

$$\operatorname{colim}_{[n]\in\mathbf{\Delta}^{\operatorname{op}}}\operatorname{Ind}_{C_n}^{\mathbb{T}}(X_n)\in D(k)^{B\mathbb{T}}$$

If we apply the \mathbb{T} -Tate construction to it, and *incorrectly* interchange $(-)^{t\mathbb{T}}$ with $\operatorname{colim}_{[n]}$, we get

$$\operatorname{colim}_{[n]\in\mathbf{\Delta}^{\operatorname{op}}} (\operatorname{Ind}_{C_n}^{\mathbb{T}}(X_n))^{t\mathbb{T}} \in D(k)$$

where $(\operatorname{Ind}_{C_n}^{\mathbb{T}}(X_n))^{t\mathbb{T}} \simeq X_n^{tC_n}[-1]$ [HN20, Prop 3], thus we see that this computes the polynomial periodic cyclic homology. Note that every *G*-module *M* gives rise to a cohomological *G*-Mackey functor $\underline{M}: [G/H] \mapsto M^H$, we can endow C_n -module X_n a cohomological C_n -Mackey functor structure, and thus realize the geometric realization $|X_{\bullet}|_{\Lambda}$ as an object of $\operatorname{Mod}_{\underline{k}}(\operatorname{Sp}^{\mathbb{T}})$, and apply the de-completed \mathbb{T} -Tate construction $(-)^{\theta_{\underline{k}}\mathbb{T}}$ to it, obtaining polynomial periodic cyclic homology.

Remark 3.1. This procedure can be made more rigorous by considering cohomological Mackey functors over Connes' cyclic category Λ^{op} . Since we do not depend on Kaledin's original construction, we skip such a development. However, a toy version of this is explained in Appendix A.

Question 1. This comparison does not compare the lax symmetric monoidal structure of Kaledin's polynomial periodic cyclic homology and ours. How do we compare the ring structure on the two?

Now we give our formal definitions, and explain why it corresponds to the construction above for associative flat k-algebras.

Construction 3.2. Let k be a t-bounded^{3.1} animated ring. The universal property of THH as in [ABG+18] gives rise to a map $\text{THH}(k) \to \underline{k}$ of $\mathbb{T}\text{-}\mathbb{E}_{\infty}\text{-}\text{rings}$. Thus for every THH(k)-module M in cyclonic spectra, we get an object $M \otimes_{\mathbb{T}\text{HH}(k)}^{\mathbb{T}} \underline{k} \in$ $\text{Mod}_{\underline{k}}(\text{Sp}^{g^{\leq}\mathbb{T}})$. In particular, let \mathcal{C} be a dualizable presentable stable k-linear ∞ category, we have a canonical genuine equivariant enhancement of the k-linear Hochschild homology of \mathcal{C} , denoted by $\text{HH}(\mathcal{C}/\underline{k}) \in \text{Mod}_{\underline{k}}(\text{Sp}^{g^{\leq}\mathbb{T}})$.

Definition 3.3. Let k be a t-bounded animated ring, and C a dualizable presentable stable k-linear ∞ -category. Then the polynomial periodic cyclic homology $\mathrm{HP^{poly}}(\mathcal{C}/k)$ (resp. the polynomial negative cyclic homology $\mathrm{HC^{-,poly}}(\mathcal{C}/k)$) is defined to be the <u>k</u>-de-completed \mathbb{T} -Tate construction $\mathrm{HH}(\mathcal{C}/\underline{k})^{\theta_{\underline{k}}\mathbb{T}}$ (resp. the <u>k</u>de-completed homotopy \mathbb{T} -fixed points $\mathrm{HH}(\mathcal{C}/\underline{k})^{\eta_{\underline{k}}\mathbb{T}}$).

^{3.1.} This means that it is bounded with respect to the canonical t-structure. We add the prefix "t-" to avoid confusion with boundedness of p-power torsion (which we do not use in this article anyways).

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Let k be a commutative ring, and R an associative flat k-algebra. Recall that, as in [ABG+18, §6], the Hochschild homology $\operatorname{HH}(R/\underline{k})$ relative to the constant Tambara functor \underline{k} is informally given by the geometric realization of relative norms $[n] \mapsto R^{\otimes_{\underline{k}}^{\mathbb{L}}C_n}$. In [Mao], we formally identify these norms with the naive tensor powers equipped with the obvious cyclic action. Thus the previous discussion basically identifies the polynomial periodic cyclic homology $\operatorname{HP}^{\operatorname{poly}}(R/k)$ with Kaledin's original one.

4. DE RHAM AND PRISMATIC COMPARISON

In this section, we will first establish a comparison result (Proposition 4.4), which says that, for smooth commutative algebras, polynomial periodic cyclic homology is the same as periodic cyclic homology. As a consequence, polynomial periodic cyclic homology of commutative algebras acquires a filtration whose associated graded pieces are equivalent to shifts of (non-Hodge-completed) derived de Rham cohomology. We then define a "topological" analogue, and for smooth algebras over perfectoid rings, it acquires a *motivic filtration* whose associated graded pieces are equivalent to shifts of (non-Nygaard-completed) prismatic cohomology.

We first give some sufficient conditions for assembly maps being equivalences.

Lemma 4.1. Let A is a commutative algebra in cyclonic spectra (resp. G-spectra for a finite group G) whose underlying Borel equivariant spectrum, denoted by A^h , is t-bounded. Then the assembly map $(-)^{\theta_A G} \to (-)^{tG}$ (resp. $(-)^{\eta_A G} \to (-)^{hG}$) is an equivalence on the idempotent-complete stable subcategory of $\operatorname{Mod}_A(\operatorname{Sp}^{g<\mathbb{T}})$ (resp. $\operatorname{Mod}_A(\operatorname{Sp}^{gG})$) generated by A-modules of the form $\bigoplus_{i\in I} A \otimes [\mathbb{T}/H_i]$ (resp. $\bigoplus_{i\in I} A \otimes [G/H_i]$) for an indexed family $(H_i)_{i\in I}$ of finite cyclic groups H_i (resp. finite groups $H_i \subseteq G$).

Proof. We write the argument for the finite group case. The cyclonic case is similar.

- By construction, the assembly map is an equivalence on $A \otimes [G/H]$ for finite groups $H \subseteq G$.
- The family $\{A^h \otimes [G/H] \in \operatorname{Sp}^{BG} | (H \subseteq G \text{ is a finite subgroup})\}$ is uniformly t-bounded, thus the canonical map

$$\bigoplus_{i} (A \otimes [G/H_i])^{hG} \longrightarrow \left(\bigoplus_{i} A \otimes [G/H_i]\right)^{hG}$$

is an equivalence for an indexed family $(H_i \subseteq G)_i$ of finite subgroups of G. It follows that the assembly map is an equivalence on $\bigoplus_i A \otimes [G/H_i]$.

• The result follows from the fact that the functors $(-)^{\theta_A G}$, $(-)^{\eta G}$, $(-)^{tG}$ and $(-)^{hG}$ are exact.

When G is a finite cyclic group, the situation is particularly simple.

Corollary 4.2. Let G be a finite cyclic group. Then the assembly map $(-)^{\theta_{\mathbb{Z}}G} \rightarrow (-)^{\theta_{\mathbb{Z}}G}$ (resp. $(-)^{\eta_{\mathbb{Z}}G} \rightarrow (-)^{\eta_{\mathbb{Z}}G})$ is an equivalence on t-bounded objects in $Mod_{\mathbb{Z}}(Sp^{gG})$.

Proof. Since the abelian category $\operatorname{Mack}_{G}^{\operatorname{coh}}$ has finite projective dimension when G is finite cyclic [BSW17, Cor 7.2], every t-bounded object is represented by a finite complex of projective objects. Thus the assembly map $(-)^{\theta G} \to (-)^{tG}$ is an equivalence on these objects by Lemma 4.1.

Corollary 4.3. Let G be a finite cyclic group, and A a commutative algebra in $D^b \operatorname{Mack}_G^{\operatorname{coh}} \subseteq \operatorname{Mod}_{\mathbb{Z}}(\operatorname{Sp}^{gG})$. Then the assembly map $(-)^{\theta_{\mathbb{Z}}G} \to (-)^{\theta_AG}$ (resp. $(-)^{\eta_{\mathbb{Z}}G} \to (-)^{\eta_AG})$ is an equivalence. Consequently, the A-de-completed G-Tate construction $(-)^{\theta_AG}$ (resp. the A-de-completed homotopy G-fixed points $(-)^{\eta_AG}$) coincides with the composite functor

$$\operatorname{Mod}_A(\operatorname{Sp}^{gG}) \longrightarrow \operatorname{Mod}_{\underline{\mathbb{Z}}}(\operatorname{Sp}^{gG}) \xrightarrow{(-)^{\theta_{\underline{\mathbb{Z}}}G} \text{ or } (-)^{\eta_{\underline{\mathbb{Z}}}G}} D(\mathbb{Z}).$$

Proof. Note that the ∞ -category $\operatorname{Mod}_A(\operatorname{Sp}^{gG})$ is generated by objects of the form $M \otimes_{\underline{\mathbb{Z}}}^{\underline{L}} A$ for finite permutation *G*-modules *M*, which is *t*-bounded since *M* is $\underline{\mathbb{Z}}$ -flat, and the result follows from Corollary 4.2.

We are ready to establish the comparison between the polynomial periodic (resp. negative) cyclic homology and the periodic (resp. negative) cyclic homology on smooth algebras:

Proposition 4.4. Let k be a t-bounded animated ring, and R an \mathbb{E}_1 -k-algebra. Then the commutative diagram

$$\begin{array}{ccc} \mathrm{HC}^{-,\mathrm{poly}}(R/k) & \longrightarrow & \mathrm{HC}^{-}(R/k) \\ & & & \downarrow \\ \mathrm{HP}^{\mathrm{poly}}(R/k) & \longrightarrow & \mathrm{HP}(R/k) \end{array}$$

as an instance of (2.1) is Cartesian. If R is p-completely smooth as an \mathbb{E}_1 -kalgebra, and have bounded Tor-amplitude in $D(k)_p^{\wedge}$, then the horizontal assembly maps are equivalences after p-completion.

We first give a proof in the special case of R being a p-completely smooth animated k-algebra (i.e. with commutativity), since the general case needs knowledge on polygonic spectra in [KMN23], and the commutative case is sufficient for this section.

Proof of Proposition 4.4 with commutativity. By Remark 2.2, this commutative diagram is Cartesian, thus for *p*-completely smooth animated *k*-algebras *R*, it suffices to check that the map $\operatorname{HP}^{\operatorname{poly}}(R/k) \to \operatorname{HP}(R/k)$ is an equivalence after modulo *p*, which is equivalent to base change along $k \to k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p$, thus we may assume that *k* is a *t*-bounded animated \mathbb{F}_p -algebra. We can further check it after modulo *p* again. By Remark 2.8, it reduces to check that the assembly map

$$\operatorname{HH}(R/k)^{\theta_{\underline{k}}C_p} \longrightarrow \operatorname{HH}(R/k)^{tC_p}$$

is an equivalence. By Corollaries 4.2 and 4.3, it suffices to check that the cyclonic spectrum $\operatorname{HH}(R/\underline{k}) \in \operatorname{Mod}_{\underline{k}}(\operatorname{Sp}^{g^{<}\mathbb{T}})$ is *t*-bounded (thus so after forgetting to $\operatorname{Sp}^{gC_{p}}$). We check in two steps.

- **Polynomial case.** When $R = P \otimes_{\mathbb{F}_p}^{\mathbb{L}} k$ where P is a (finite) polynomial \mathbb{F}_p algebra. Then we have $\operatorname{HH}(R/\underline{k}) = \operatorname{HH}(P/\underline{\mathbb{F}_p}) \otimes_{\mathbb{F}_p}^{\mathbb{L}} k$, and by *t*-boundedness
 of k, it suffices to show that $\operatorname{HH}(P/\underline{\mathbb{F}_p})$ is *t*-bounded. This follows from
 [Hes96, 2.2.4 & 2.2.5].
- **General case.** By passing to a Zariski cover, we may assume that there exists an étale map $S \to R$ where S is a (finite) polynomial k-algebra. Then by [HLL20, Add 3.2] (along with [Bor11, Thm B], which is used in their proof), the map $\text{THH}(S) \to \text{THH}(R)$ is flat (even étale) in $\text{CAlg}(\text{Sp}^{g < \mathbb{T}})$, thus so is the map $\text{HH}(S/\underline{k}) \to \text{HH}(R/\underline{k})$, and the result follows.

Now we give the proof for Proposition 4.4 in full generality. As in the first proof of Proposition 4.4, it reduces to check that the assembly map

$$\operatorname{HH}(R/\underline{k})^{\theta_{\underline{k}}C_p} \longrightarrow \operatorname{HH}(R/k)^{tC_p}$$

is an equivalence when k is a t-bounded animated \mathbb{F}_p -algebra, and R is p-completely smooth as an \mathbb{E}_1 -k-algebra with bounded Tor-amplitude in D(k). We prove a generalized version with coefficients.

Let R be an \mathbb{E}_1 -ring spectrum, and M an R-R-bimodule in Sp. Then by [KMN23, §6], the topological Hochschild homology THH(R; M) carries a canonical p-polygonic structure, given by the sequence $(\text{THH}(R; M^{\otimes_{RP}^{E}}) \in \text{Sp}^{BC_pr})_{r \in \mathbb{N}}$, along with polygonic Frobenius maps

$$\operatorname{THH}(R; M^{\otimes_R^{\mathbb{Z}} p^r}) \longrightarrow \operatorname{THH}(R; M^{\otimes_R^{\mathbb{Z}} p^{r+1}})^{tC_p}$$

in $\operatorname{Sp}^{BC_{p^r}}$. Moreover, when all $\operatorname{THH}(R; M^{\otimes_{Rp^r}})$ in question are bounded below (this is the case when M is a perfect R-R-bimodule in Sp), we get a sequence $(\operatorname{THH}(R; M^{\otimes_{Rp^r}}) \in \operatorname{Sp}^{gC_{p^r}})_{r \in \mathbb{N}}$ with equivalences

$$\operatorname{THH}(R; M^{\otimes_{R}^{\mathbb{L}}p^{r+1}})^{\Phi C_p} \xrightarrow{\simeq} \operatorname{THH}(R; M^{\otimes_{R}^{\mathbb{L}}p^{r}})$$

of genuine C_{p^r} -spectra. This construction extends to the case without boundedbelow-ness, by a forthcoming work by Harpaz–Nikolaus–Saunier. All of these constructions are functorial in (R; M).

There is also a forgetful functor from cyclotomic spectra to polygonic spectra, and THH(R; R) as a polygonic spectrum is the same as the underlying polygonic spectrum of the cyclotomic spectrum THH(R).

Construction 4.5. Let k be an animated ring, R an \mathbb{E}_1 -k-algebra, and M an R-Rbimodule in D(k). Then for every $r \in \mathbb{N}$, we get a genuine equivariant enhancement $\operatorname{HH}((R; M)/\underline{k})$ of Hochschild homology $\operatorname{HH}((R; M^{\otimes_R^{\mathbb{E}} p^r})/k)$ with coefficients given by

$$\operatorname{HH}((R; M^{\otimes_{R}^{\mathbb{Z}}p^{r}}) / \underline{k}) := \operatorname{THH}(R; M^{\otimes_{R}^{\mathbb{Z}}p^{r}}) \otimes_{\operatorname{THH}(k)}^{\mathbb{L}} \underline{k}.$$

Lemma 4.6. Let k be a t-bounded animated ring, R an \mathbb{E}_1 -k-algebra with bounded Tor-amplitude in D(k), and M a perfect R-R-bimodule in D(k). Then the assembly map

$$\operatorname{HH}((R; M^{\otimes_{R}^{\mathbb{L}}p}) / \underline{k})^{\theta_{\underline{k}}C_{p}} \longrightarrow \operatorname{HH}((R; M^{\otimes_{R}^{\mathbb{L}}p}) / k)^{tC_{p}}$$

is an equivalence.

Proof. The proof of [NS18, Prop III.1.1] (or [Lur11, Prop 2.2.3]) implies that the functor $\operatorname{HH}((R; (-)^{\otimes_{\overline{R}}\mathbb{P}})/k)^{tC_p}$ is exact. By Lemma 2.10, the symmetric monoidal structure on $(-)^{\Phi C_p}$, and the equivalence $\operatorname{THH}(R; M^{\otimes_{\overline{R}}\mathbb{P}})^{\Phi C_p} \xrightarrow{\simeq} M$, we see that the functor $\operatorname{HH}((R; (-)^{\otimes_{\overline{R}}\mathbb{P}})/\underline{k})^{\Phi C_p}$ is exact as well. Thus, to see that the assembly map in question is an equivalence, it suffices to show that it is an equivalence when $M = R \otimes_{k}^{\mathbb{L}} R$, the free *R-R*-bimodule in D(k) of rank 1. In this case, we have an equivalence

$$\operatorname{THH}(R; (R \otimes_{k}^{\mathbb{L}} R)^{\otimes_{R}^{\mathbb{L}} p^{r}}) \simeq \operatorname{THH}(k; R^{\otimes_{k}^{\mathbb{L}} p^{r}})$$

in $\operatorname{Mod}_{\operatorname{THH}(k)}(\operatorname{Sp}^{gC_pr})$, and after base change along $\operatorname{THH}(k) \to \underline{k}$, it becomes the relative C_{p^r} -norm $R^{\otimes_{\underline{k}} C_pr}$. Under this identification, the assembly map in question becomes the assembly map

$$\left(R^{\otimes \underline{\mathbb{L}} C_{p^r}} \right)^{\theta_{\underline{k}} C_{p^r}} \longrightarrow \left(R^{\otimes \underline{\mathbb{L}} C_{p^r}} \right)^{t C_{p^r}}$$

Note that the derived cohomological C_{p^r} -Mackey functor $R^{\otimes_k^{\mathbb{L}} C_{p^r}}$ is bounded, by the Tor-amplitude boundedness of R in D(k), and the *t*-boundedness of k. The result follows from Corollaries 4.2 and 4.3.

Proof of Proposition 4.4 in general. It follows from Lemma 4.6 by setting M = R. \Box

Question 2. Is there any categorical generalization of Lemma 4.6, namely, replacing R by dualizable presentable stable k-linear ∞ -category \mathcal{C} which is smooth with bounded Tor-amplitude^{4.1}, and replacing M by an colimit-preserving k-linear end-ofunctor $\mathcal{C} \to \mathcal{C}$?

Now we construct an HKR filtration on the polynomial periodic (resp. negative) cyclic homology whose associated graded pieces are shifts of (non-Hodgecompleted) derived de Rham cohomology (resp. its Hodge-filtered pieces), Hodgede-completing [BMS19, Thm 1.17] and [Ant19].

Construction 4.7. (HKR filtration) Let k be a t-bounded animated ring. Then by [Rak20] (which generalizes [Ant19]), for every smooth k-algebra R, there is an exhaustive filtration Fil^{*}_{HKR}, functorial in R, on the canonical p-completed map

$$\operatorname{HC}^{-}(R/k)_{p}^{\wedge} \longrightarrow \operatorname{HP}(R/k)_{p}^{\wedge}$$

whose *i*-th associated graded piece $\operatorname{gr}_{HKR}^{i}$ is given by the canonical map

$$\operatorname{Fil}_{H}^{i} \mathrm{dR}_{R/k}[2\,i]_{p}^{\wedge} \to \mathrm{dR}_{R/k}[2\,i]_{p}^{\wedge}.$$

By Proposition 4.4 and sifted-colimit-preservation of the functors $\operatorname{CAlg}_k^{\operatorname{an}} \to D(k)$, $R \mapsto \operatorname{HP}^{\operatorname{poly}}(R/k)$ (resp. $R \mapsto \operatorname{HC}^{-,\operatorname{poly}}(R/k)$), for every animated k-algebra R, we get an exhaustive filtration $\operatorname{Fil}_{\operatorname{HKR}}^*$ on the Cartesian square

$$\begin{array}{cccc} \mathrm{HC}^{-,\mathrm{poly}}(R/k)_p^{\wedge} &\longrightarrow & \mathrm{HC}^{-}(R/k)_p^{\wedge} \\ & & & \downarrow \\ \mathrm{HP}^{\mathrm{poly}}(R/k)_p^{\wedge} &\longrightarrow & \mathrm{HP}(R/k)_p^{\wedge} \end{array}$$

whose *i*-th associated graded piece gr_{HKR}^{i} is given by

$$\begin{array}{ccc} \operatorname{Fil}_{H}^{i} \operatorname{dR}_{R/k}[2\,i]_{p}^{\wedge} &\longrightarrow & \operatorname{Fil}_{H}^{i} \operatorname{dR}_{R/k}[2\,i]_{p}^{\wedge} \\ & & \downarrow \\ & & \downarrow \\ \operatorname{dR}_{R/k}[2\,i]_{p}^{\wedge} &\longrightarrow & \operatorname{dR}_{R/k}[2\,i]_{p}^{\wedge} \end{array}$$

where $\widehat{dR}_{R/k}$ is the Hodge-completed derived de Rham cohomology of R/k.

Question 3. Is the HKR filtration in Construction 4.7 complete?

It is very natural to extend our definition to de-Nygaard-complete topological periodic cyclic homology.

^{4.1.} An attempt for this definition: a dualizable presentable stable k-linear ∞ -category C has bounded Tor-amplitude if the coevaluation functor $C^{\vee} \otimes_{D(k)} C \to D(k)$ sends compact objects to objects of bounded Tor-amplitude in D(k).

Definition 4.8. Let k be a t-bounded animated ring, and C a dualizable presentable stable k-linear ∞ -category. Then the topological k-polynomial periodic cyclic homology $\mathrm{TP}^{\mathrm{poly}/k}(\mathcal{C})$ (resp. the topological k-polynomial negative cyclic homology $\mathrm{TC}^{-,\mathrm{poly}/k}(\mathcal{C})$) is defined to be the $\mathrm{THH}(k)$ -de-completed \mathbb{T} -Tate construction $\mathrm{THH}(\mathcal{C})^{\theta_{\mathrm{THH}(k)}\mathbb{T}}$ (resp. the $\mathrm{THH}(k)$ -de-completed homotopy \mathbb{T} -fixed points $\mathrm{THH}(\mathcal{C})^{\eta_{\mathrm{THH}(k)}T}$).

Topological k-polynomial periodic (resp. negative) cyclic homology, even after p-completion, seems intractable in general, partially due to the global nature of its prismatization. The situation is drastically simpler when the ring k = S is p-complete and perfectoid, thanks to the Bökstedt periodicity of $\text{THH}(S)_p^{\wedge}$. We recollect some notations and computations in [BMS19, Prop 6.2 & 6.3]. However, we view $\text{THH}(S)^{tC_p}$ non-equivariantly as a $\text{TC}^-(S)$ -module, which follows more closely to the convention in [Rig22, Lem 2.1].

Remark 4.9. ([BMS19, Prop 6.2 & 6.3]) Let S be a perfectoid ring, $A := A_{inf}(S) = W(S^{\flat})$ with Frobenius endomorphism $\varphi: A \to A$, and ξ a chosen generator of the kernel ker(θ) of Fontaine's map $\theta: A \to S$. Then the commutative square

$$\begin{array}{ccc} \mathrm{TC}^{-}(S)_{p}^{\wedge} & \xrightarrow{\varphi_{p}^{h\mathrm{T}}} & \mathrm{TP}(S)_{p}^{\wedge} \\ & & \downarrow \\ \mathrm{THH}(S)_{p}^{\wedge} & \xrightarrow{\varphi_{p}} & \mathrm{THH}(S)^{tC_{p}} \end{array}$$

is a pushout diagram of \mathbb{E}_{∞} -rings, and its homotopy groups are given by

$$\begin{array}{ccc} A[u,v]/(u\,v-\xi) & \xrightarrow{u\mapsto\sigma} & A[\sigma^{\pm}] \\ & & \downarrow & & \downarrow \\ R[u] = (A/\xi)[u] & \xrightarrow{u\mapsto\sigma} & (A/\varphi(\xi))[u] \end{array}$$

where $|u| = |\sigma| = 2$ and |v| = -2, the vertical maps are A-linear, and the horizontal maps are φ -linear. The homotopy groups of the canonical map $\mathrm{TC}^{-}(S)_{p}^{\wedge} \to \mathrm{TP}(S)_{p}^{\wedge}$ is given by the A-linear map

$$A[u,v]/(uv-\xi) \xrightarrow[v \mapsto \sigma^{-1}]{u \mapsto \xi\sigma} A[\sigma^{\pm}].$$

In particular, the $\mathrm{THH}(S)\mathrm{-module}\ S=\mathrm{THH}(S)\,/\,u$ in $\mathrm{Sp}^{B\mathbb{T}}$ is perfect, which implies that

Lemma 4.10. Let S be a perfectoid ring, and M a THH(S)-module in $\text{Sp}^{g \leq \mathbb{T}}$. Then the canonical map

$$M^{\theta_{\mathrm{THH}(S)}\mathbb{T}} \otimes_{\mathrm{TP}(S)}^{\mathbb{L}} S^{t\mathbb{T}} \longrightarrow (M \otimes_{\mathrm{THH}(S)} S)^{\theta_S \mathbb{T}}$$

is an equivalence after p-completion.

Corollary 4.11. Let S be a perfectoid ring, and C a dualizable presentable stable S-linear ∞ -category. Then the commutative diagram

$$\begin{array}{ccc} \mathrm{TC}^{-,\mathrm{poly}/S}(\mathcal{C}) & \longrightarrow & \mathrm{TC}^{-}(\mathcal{C}) \\ & & & \downarrow \\ & & & \downarrow \\ \mathrm{TP}^{\mathrm{poly}/S}(\mathcal{C}) & \longrightarrow & \mathrm{TP}(\mathcal{C}) \end{array}$$

as an instance of (2.1) is Cartesian. If the assembly map $\operatorname{HPpoly}(\mathcal{C}/S) \to \operatorname{HP}(\mathcal{C}/S)$ is an equivalence after p-completion^{4.2}, then the horizontal assembly maps are equivalences after $(p, \operatorname{ker}(\theta))$ -completion.

Proof. By Remark 2.2, this commutative diagram is Cartesian. When the assembly map $\operatorname{HP}^{\operatorname{poly}}(\mathcal{C}/S) \to \operatorname{HP}(\mathcal{C}/S)$ is *p*-completely an equivalence, then by Remark 4.9, it is the $(\operatorname{mod} \ker(\theta))$ reduction of the map $\operatorname{TP}^{\operatorname{poly}/S}(\mathcal{C}) \to \operatorname{TP}(\mathcal{C})$, thus the later is $(p, \ker(\theta))$ -completely an equivalence, and the result follows.

Similarly to Construction 4.7, we have

Construction 4.12. (Motivic filtration) Let S be a perfectoid ring, and let $A := A_{inf}(S)$. Then by [BMS19], for every smooth S-algebra R, there is an exhaustive filtration Fil^{*}_M, functorial in R, on the canonical $(p, \ker(\theta))$ -completed^{4.3} map

$$\mathrm{TC}^{-}(R)^{\wedge}_{(p,\ker(\theta))} \longrightarrow \mathrm{TP}(R)^{\wedge}_{(p,\ker(\theta))}$$

whose *i*-th associated graded piece gr_M^i is given by the canonical map

$$\operatorname{Fil}_{N}^{i}\varphi_{A}^{*} \boxtimes_{R/A} \longrightarrow \varphi_{A}^{*} \boxtimes_{R/A}$$

for the Frobenius twisted prismatic cohomology. By Proposition 4.4 and Corollary 4.11, and sifted-colimit-preservation of the functors $\operatorname{CAlg}_S^{\operatorname{an}} \to D(\operatorname{TC}^-(S))$, $R \mapsto \operatorname{TP}^{\operatorname{poly}/S}(R)$ (resp. $R \mapsto \operatorname{TC}^{-,\operatorname{poly}/S}(R)$), for every animated k-algebra R, we get an exhaustive filtration Fil_M^* on the Cartesian square

$$\begin{array}{cccc} \mathrm{TC}^{-,\mathrm{poly}/S}(R)^{\wedge}_{(p,\ker(\theta))} &\longrightarrow & \mathrm{TC}^{-}(R)^{\wedge}_{(p,\ker(\theta))} \\ & & \downarrow & & \downarrow \\ \mathrm{TP}^{\mathrm{poly}/S}(R)^{\wedge}_{(p,\ker(\theta))} &\longrightarrow & \mathrm{TP}(R)^{\wedge}_{(p,\ker(\theta))} \end{array}$$

whose *i*-th associated graded piece is given by

$$\begin{array}{cccc} \operatorname{Fil}_{N}^{i} \varphi_{A}^{*} \, \underline{\mathbb{A}}_{R/A} & \longrightarrow & \operatorname{Fil}_{N}^{i} \varphi_{A}^{*} \, \underline{\mathbb{A}}_{R/A} \\ & & \downarrow & & \downarrow \\ \varphi_{A}^{*} \, \underline{\mathbb{A}}_{R/A} & \longrightarrow & \varphi_{A}^{*} \, \underline{\mathbb{A}}_{R/A} \end{array}$$

Remark 4.13. The construction of HKR filtration and motivic filtration on polynomial cyclic theories is quite formal: one only needs a proposition similar to Proposition 4.4 and Corollary 4.11. In particular, the above construction also adapts to the Breuil–Kisin case (as in [BMS19, §11]) and the q-de Rham case.

Remark 4.14. The topological polynomial periodic cyclic homology should be comparable to $TC^{(-1)}$ as introduced in [Man24, §1], or some construction in an ongoing project of Devalapurkar–Hahn–Raksit–Yuan as mentioned in [Man24, Rem 1.6].

A natural question is whether the previous picture extends to relative prismatic cohomology over an arbitrary base prism? When the base prism is transveral, we have the following expectation.

^{4.2.} By Proposition 4.4, this is the case when $\mathcal{C} = D(R)$ for some *p*-completely smooth \mathbb{E}_1 -S-algebra R.

^{4.3.} The *p*-completed $\mathrm{TC}^-(R)_p^{\wedge}$ and $\mathrm{TP}(R)_p^{\wedge}$ are automatically ker(θ)-complete (since they are Nygaard-complete), but it is conceptually better to phrase it after $(p, \ker(\theta))$ -completion, since the *p*-completed polynomial versions might not be ker(θ)-complete.

Remark 4.15. In a companion paper [Mao24], we defined prismatic Hochschild homology $\operatorname{HH}^{\mathbb{A}}(\mathcal{C}/A)$ for a transversal prism (A, I) and a dualizable presentable stable A/I-linear ∞ -category \mathcal{C} , and formulated an HKR-type conjecture for p-completely smooth A/I-algebras. If that conjecture holds, then the p-completed $(\underline{A}, I)^{C_{p^{r-1}}}$ -de-completed $\mathbb{T}/C_{p^{r-1}}$ -construction ($\operatorname{HH}^{\mathbb{A}}(R/A)^{C_{p^{r-1}}}$) $\stackrel{\theta}{\xrightarrow{(A,I)}} \stackrel{C_{p^{r-1}}(\mathbb{T}/C_{p^{r-1}})}{\operatorname{for animated}}$ for animated (A/I)-algebra would carry an exhaus-

 $A)^{\mathbb{C}_{p^{r-1}}}$ for animated (A/I)-algebra would carry an exhaustive filtration with associated graded pieces equivalent to shifts of $(\varphi_{A}^{*} \bigtriangleup_{R/A}) \otimes_{A}^{\mathbb{L}} (A/I_{r})$. We will address this in the future.

Up to our knowledge, it was not widely expected that the topological Hochschild homology as a cyclotomic spectrum contains enough information for a de-Nygaardcompletion such as Definition 4.8. However, in view of Efimov's rigidity of localizing motives, this is expected:

Remark 4.16. (M. RAMZI) Let k be a commutative ring, and G a finite group, A an \mathbb{E}_{∞} -algebra in Sp^{gG} , and $E: \operatorname{Cat}_{k}^{\operatorname{perf}} \to \operatorname{Mod}_{A}(\operatorname{Sp}^{BG})$ a finitary symmetric monoidal localizing invariant. Then the symmetric monoidal functor E factors uniquely through the presentably stable symmetric monoidal ∞ -category $\operatorname{Mot}_{\operatorname{loc},k}$, obtaining a functor $\operatorname{Mot}_{\operatorname{loc},k} \to \operatorname{Mod}_{A}(D(k)^{BG})$ in $\operatorname{CAlg}(\operatorname{Pr}_{\operatorname{St}}^{L})$. Efimov's rigidity theorem, as mentioned in [Efi24, Rem 4.3], tells us that the presentably stable symmetric monoidal ∞ -category $\operatorname{Mot}_{\operatorname{loc},k}$ is rigid, thus we get a unique strongly continuous functor $\operatorname{Mot}_{\operatorname{loc},k} \to \operatorname{Mod}_{A}(D(k)^{BG})^{\operatorname{rig}}$.

If we are in addition given a finitary symmetric monoidal factorization

$$\operatorname{Cat}_{k}^{\operatorname{perf}} \xrightarrow{E} \operatorname{Mod}_{A}(\operatorname{Sp}^{gG}) \longrightarrow \operatorname{Mod}_{A}(D(k)^{BG})$$

of E, where the functor \tilde{E} is localizing as well, then by Remark 2.7, we see that the strongly continuous functor $\operatorname{Mot}_{\operatorname{loc},k} \to \operatorname{Mod}_A(D(k)^{BG})^{\operatorname{rig}}$ coincides with the composite functor

$$\operatorname{Mot}_{\operatorname{loc},k} \xrightarrow{\tilde{E}} \operatorname{Mod}_A(\operatorname{Sp}^{gG}) \longrightarrow \operatorname{Mod}_A(D(k)^{BG})^{\operatorname{rig}}.$$

Informally, this tells us that \tilde{E} "knows" everything about refined E.

Now we apply this to the finitary symmetric monoidal localizing invariant

$$\tilde{E} := \operatorname{THH} : \operatorname{Cat}_{k}^{\operatorname{perf}} \longrightarrow \operatorname{Cyc}\operatorname{Sp^{gen}} \longrightarrow \operatorname{Mod}_{\operatorname{THH}(k)}(\operatorname{Sp}^{gG})$$

for any finite cyclic group G. It follows that the refinement of Sp^{BG} -valued THH is completely determined by the functor \tilde{E} , thus by the (genuine) cyclotomic THH. Roughly speaking, this implies that the cyclotomic THH already contains "all" profinite or *p*-adic formal information, including any de-Nygaard-completion.

5. Comparison to THH

Recall that the *Cartier isomorphism* identifies algebraic de Rham cohomology groups of smooth \mathbb{F}_p -algebras with their algebraic differential forms. We give two noncommutative analogues. In this section, we discuss one of them, which compares polynomial coperiodic cyclic homology of dualizable presentable stable \mathbb{F}_p -linear ∞ -categories with their topological Hochschild homology. This comparison was proved in [Kal20, Cor 11.15] for associative \mathbb{F}_p -algebras, using Goodwillie derivative of (Hochschild–)Witt trace theory. We give two arguments. Although the second argument is much shorter, the first argument gives us more information, which is used in Section 6. The key to the first argument is the observation that the \mathbb{T}/C_p -equivariant \mathbb{E}_1 - \mathbb{Z}^{tC_p} -module $\mathbb{F}_p^{tC_p}$ is co-induced.

Construction 5.1. Let $n \in \mathbb{N}_{>0}$. There is a map

$$\mathbb{Z}^{\mathbb{T}/C_n} \longrightarrow \mathbb{Z}/n$$

in $D(\mathbb{Z})^{B\mathbb{T}}$, where \mathbb{Z}/n is equipped with the trivial action, i.e. $\varpi_1^*(\mathbb{Z}/n)$. Indeed, this map is taken to be represented^{5.1} by the following surjective map

$$\begin{array}{cccc} 0 & & -1 \\ \mathbb{Z} & \stackrel{0}{\longleftrightarrow} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}/n & \stackrel{0}{\longleftrightarrow} & 0 \end{array}$$

of mixed complexes concentrated in homological degrees [-1,0]. Since it is surjective, it is a fibration in the projective model structure on mixed complexes, with kernel being the mixed complex $\left(n \mathbb{Z} \stackrel{0}{\underset{n}{\longleftrightarrow}} \mathbb{Z}\right)$, which represents $\mathbb{Z}^{\mathbb{T}} \in D(\mathbb{Z})^{B\mathbb{T}}$. Consequently, we get a fiber sequence

$$\mathbb{Z}^{\mathbb{T}} \longrightarrow \mathbb{Z}^{\mathbb{T}/C_n} \longrightarrow \mathbb{Z}/n$$

in $D(\mathbb{Z})^{B\mathbb{T}}$.

Remark 5.2. One can also construct the map and the fiber sequence in Construction 5.1 by the gold relation $a_{\lambda^n} u_{\lambda^m} = (n/m) a_{\lambda^m} u_{\lambda^n}$ for $(m, n) \in \mathbb{N}^2_{>0}$ with $m \mid n$ in [HHR17, Lem 3.6] (where u_{λ^m} 's become equivalence after forgetting to the Borel equivariant objects), where λ^n is the complex S^1 -representation $S^1 \xrightarrow{(-)^n} \mathbb{C}^{\times}$ viewed as a real representation, and a_{λ^n} is the Euler class $\mathbb{S}^0 \to \mathbb{S}^{\lambda^n}$, i.e. the map of Thom spectra of the inclusion $\{0\} \subseteq \lambda^n$ of representations. Indeed, there is a fiber sequence

$$\Sigma^{\infty}_{\mathbb{T}} [\mathbb{T} / C_n]_+ \longrightarrow \mathbb{S}^0 \longrightarrow \mathbb{S}^{\lambda^r}$$

of cyclonic spectra (and thus, of \mathbb{T} -equivariant spectra), cf. [Sul20, Obs 2.32]. Taking \mathbb{Z} -linear dual, and applying the "octahedral axiom" to the gold relation $a_{\lambda^n} \sim (n/m) a_{\lambda^m}$, we get the fiber sequence in Construction 5.1.

This allows us to establish results beyond char p, as observed by Yuri SULYMA in [Sul20, Lem 4.9].

Corollary 5.3. The map $\mathbb{Z}^{\mathbb{T}/C_n} \to \mathbb{Z}/n$ in $D(\mathbb{Z})^{B\mathbb{T}}$ as in Construction 5.1 induces an equivalence

$$\mathbb{Z}^{tC_n} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}^{\mathbb{T}/C_n} \longrightarrow (\mathbb{Z}/n)^{tC_n}$$

in $\operatorname{Mod}_{\mathbb{Z}^{t}C_{n}}(D(\mathbb{Z})^{B(\mathbb{T}/C_{n})}).$

Proof. Applying $(-)^{tC_n}$ to the fiber sequence in Construction 5.1, we see that the induced map $(\mathbb{Z}^{\mathbb{T}/C_n})^{tC_n} \to (\mathbb{Z}/n)^{tC_n}$ in $D(\mathbb{Z})^{B(\mathbb{T}/C_n)}$ is an equivalence (which even has an \mathbb{E}_1 -structure). It suffices to establish an equivalence

$$\mathbb{Z}^{tC_n} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}^{\mathbb{T}/C_n} \longrightarrow (\mathbb{Z}^{\mathbb{T}/C_n})^{tC_n}$$

^{5.1.} The \mathbb{E}_1 -monoidal equivalence of the monoidal ∞ -category $D(\mathbb{Z})^{B_T}$ and the monoidal ∞ -category of mixed complexes up to quasi-isomorphism is explained in details in [Lei22, §5.4].

in $\operatorname{Mod}_{\mathbb{Z}^{tC_n}}(D(\mathbb{Z})^{B(\mathbb{T}/C_n)})$. This is given by equivalences

$$\mathbb{Z}^{tC_n} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}^{\mathbb{T}/C_n} \simeq (\mathbb{Z}^{tC_n} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}[\mathbb{T}/C_n])[-1]$$
$$\simeq (\mathbb{Z}^{tC_n} \otimes (\mathbb{T}/C_n))[-1]$$
$$\simeq (\mathbb{Z}[\mathbb{T}/C_n])^{tC_n}[-1]$$
$$\simeq (\mathbb{Z}^{\mathbb{T}/C_n})^{tC_n}$$

in $\operatorname{Mod}_{\mathbb{Z}^{tC_n}}(D(\mathbb{Z})^{B(\mathbb{T}/C_n)}).$

Warning 5.4. The map in Corollary 5.3 does not carry any \mathbb{E}_1 -structure when n=2. Indeed, on the left hand side, the square-zero class $e \in \pi_{-1}(\mathbb{Z}^{\mathbb{T}/C_n})$ gives rise to a square-zero class $e \in \pi_{-1}(\mathbb{Z}^{\mathbb{T}/C_n} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}^{tC_n})$, while the homotopy ring $\pi_*(\mathbb{F}_2^{tC_2})$ is isomorphic to $\mathbb{F}_2((s))$ for a generator $s \in \pi_1(\mathbb{F}_2^{tC_2})$, which is integral, thus any square-zero elements is necessarily zero.

Question 4. Is there any version of \mathbb{E}_1 -enhancement of Corollary 5.3?

Now we compare polynomial periodic cyclic homology of dualizable presentable stable \mathbb{F}_p -linear ∞ -categories. Slightly more generally, we consider $\text{THH}(\mathbb{F}_p)$ -modules in cyclonic spectra. By Lemma 2.13 and the symmetric monoidal structure on $(-)^{\Phi C_p}$, we have

Lemma 5.5. Let k be a commutative algebra. Then the composite functor

$$\operatorname{Mod}_{\operatorname{THH}(k)}(\operatorname{Sp}^{g^{<}\mathbb{T}}) \xrightarrow{(-)\otimes^{\mathbb{L}}_{\operatorname{THH}(k)\underline{k}}} \operatorname{Mod}_{\underline{k}}(\operatorname{Sp}^{g^{<}\mathbb{T}}) \xrightarrow{((-)^{\theta_{\underline{k}}^{-}\mathbb{T}})_{p}^{\wedge}} D(k^{t^{}\mathbb{T}})_{p}^{\wedge}$$

is equivalent to the composite functor

where the vertical arrow is the base change along the map $\text{THH}(k)^{\Phi C_p} \rightarrow k^{tC_p}$ of \mathbb{T}/C_p - \mathbb{E}_{∞} -rings.

Remark 5.6. Recall that both maps in the composite map $\underline{\mathbb{Z}} \to \underline{\mathbb{Z}}_p = \operatorname{TR}(\mathbb{F}_p) \to \operatorname{THH}(\mathbb{F}_p)$ of $\mathbb{T}\text{-}\mathbb{E}_{\infty}\text{-rings}$ become equivalences after taking geometric fixed points $(-)^{\Phi C_p}$ (cf. [AMR21, Rem 10.9]). It follows that we can identify the $\mathbb{T}\text{-}\mathbb{E}_{\infty}\text{-ring}$ THH (\mathbb{F}_p) with $\underline{\mathbb{Z}}^{\Phi C_p}$. Moreover, by [HM97], there exists a Bökstedt element $\sigma \in \pi_2 \operatorname{TF}(\mathbb{F}_p)$ which maps to a Bökstedt element in $\pi_2 \operatorname{TR}^r(\mathbb{F}_p)$ for every $r \in \mathbb{N}_{>0}$, and by [NS18, Lem II.6.1] and computational results about $\operatorname{THH}(\mathbb{F}_p)$, we can identify the $\mathbb{T}\text{-}\mathbb{E}_{\infty}\text{-ring} \operatorname{THH}(\mathbb{F}_p)[\sigma^{-1}]$ with the Borel $\mathbb{T}\text{-}\mathbb{E}_{\infty}\text{-rings is simply inverting } \sigma$.

Lemma 5.7. The composite functor

$$\operatorname{Mod}_{\operatorname{THH}(\mathbb{F}_p)^{tC_p}}(\operatorname{Sp}^{g_p(\mathbb{T}/C_p)}) \longrightarrow \operatorname{Mod}_{\mathbb{F}_p^{tC_p}}(\operatorname{Sp}^{g_p(\mathbb{T}/C_p)}) \xrightarrow{(-)^{\eta_{\mathbb{F}_p^{tC_p}(\mathbb{T}/C_p)}}} D(\mathbb{Z}^{t\mathbb{T}})$$

coincides with the forgetful functor, where the first functor is the base change along the map $\text{THH}(\mathbb{F}_p)^{tC_p} \to \mathbb{F}_p^{tC_p}$ of Borel \mathbb{E}_{∞} -p-cyclonic spectra.

Recall that the map $\mathbb{Z} \to \text{THH}(\mathbb{F}_p)$ as in Remark 5.6 also becomes an equivalence after taking $(-)^{tC_p}$ by [NS18, Cor IV.4.13], thus we can replace $\text{THH}(\mathbb{F}_p)^{tC_p}$ by \mathbb{Z}^{tC_p} in Lemma 5.7.

Proof. Since both functors in question preserve filtered colimits, it suffices to check on compact objects of $\operatorname{Mod}_{\mathbb{Z}^{tC_p}}(\operatorname{Sp}^{g_p(\mathbb{T}/C_p)})$. Note that the first base change functor preserves compact objects. The result follows from Corollary 5.3.

Remark 5.8. The target of the composite functor in Lemma 5.7 has an $\mathbb{F}_p^{t\mathbb{T}}$ -module structure, and such a structure is yet to explore.

Summarizing the above discussion, we get

Proposition 5.9. The composite functor

$$\operatorname{Mod}_{\operatorname{THH}(\mathbb{F}_p)}(\operatorname{Sp}^{g^{<}\mathbb{T}}) \xrightarrow{(-)\otimes_{\operatorname{THH}(\mathbb{F}_p)}^{\mathbb{L}} \underline{\mathbb{F}_p}} \operatorname{Mod}_{\underline{\mathbb{F}_p}}(\operatorname{Sp}^{g^{<}\mathbb{T}}) \xrightarrow{(-)^{\theta_{\underline{\mathbb{F}_p}}^{\mathbb{T}}}} D(\mathbb{Z}^{t^{\mathbb{T}}})$$

coincides with with the composite functor

$$\operatorname{Mod}_{\operatorname{THH}(\mathbb{F}_p)}(\operatorname{Sp}^{g^{<}\mathbb{T}}) \xrightarrow{(-)^{\Phi C_p}} \operatorname{Mod}_{\operatorname{THH}(\mathbb{F}_p)^{\Phi C_p}}(\operatorname{Sp}^{g_p(\mathbb{T}/C_p)}) \longrightarrow D(\mathbb{Z}^{t\mathbb{T}}),$$

where the second functor is inverting σ to the underlying THH(\mathbb{F}_p) $^{\Phi C_p}$ -module.

Applying to the $\text{THH}(\mathbb{F}_p)$ -module spectrum $\text{THH}(\mathcal{C})$, we get:

Corollary 5.10. Let C be a dualizable presentable stable \mathbb{F}_p -linear ∞ -category. Then the polynomial periodic cyclic homology $\operatorname{HP}^{\operatorname{poly}}(C/\mathbb{F}_p)$ is equivalent to $\operatorname{THH}(C)[\sigma^{-1}]$ as $\mathbb{Z}^{t\mathbb{T}}$ -module spectra.

Remark 5.11. We do not compare the multiplicative structures on polynomial periodic cyclic homology (as a lax symmetric monoidal functor) with topological Hochschild homology in Corollary 5.10.

Now we give a second proof of Lemma 5.7, inspired by the proof of [Mat20, Prop 2.15]. The key is the following lemma.

Lemma 5.12. The composite functor

$$\operatorname{Mod}_{\operatorname{THH}(\mathbb{F}_p)^{tC_p}}(\operatorname{Sp}^{B(\mathbb{T}/C_p)}) \longrightarrow \operatorname{Mod}_{\mathbb{F}_p^{tC_p}}(\operatorname{Sp}^{B(\mathbb{T}/C_p)}) \xrightarrow{(-)^{h(\mathbb{T}/C_p)}} D(\operatorname{TP}(\mathbb{F}_p))$$

coincides with the forgetful functor.

Proof. Since the image of $u \in \pi_2 \operatorname{TC}^-(\mathbb{F}_p)$ under the canonical map $\operatorname{TC}^-(\mathbb{F}_p) \to \operatorname{TP}(\mathbb{F}_p)$ is $u \, p \in \pi_2 \operatorname{TP}(\mathbb{F}_p)$ as reviewed in Remark 4.9, this composite functor coincides with the composite functor

$$\operatorname{Mod}_{\operatorname{THH}(\mathbb{F}_p)^{tC_p}}(\operatorname{Sp}^{B(\mathbb{T}/C_p)}) \xrightarrow{(-)^{h(\mathbb{T}/C_p)}} D(\operatorname{TP}(\mathbb{F}_p)) \xrightarrow{(-)/\mathbb{L}_p} D(\operatorname{TP}(\mathbb{F}_p)),$$

which is subsequently identified with the forgetful functor by the proof of [BMS19, Prop 6.4] (or more precisely, the first displayed formula there).

Lemma 5.7 follows from the fact that the composite functor in Lemma 5.12, by virtue of identification with the forgetful functor, preserves filtered colimits, and that the first functor $\operatorname{Mod}_{\operatorname{THH}(\mathbb{F}_p)^{tC_p}}(\operatorname{Sp}^{g_p(\mathbb{T}/C_p)}) \to \operatorname{Mod}_{\mathbb{F}_p^{tC_p}}(\operatorname{Sp}^{g_p(\mathbb{T}/C_p)})$ in Lemma 5.7 preserves compact objects.

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6. Conjugate filtration

As explained in the introduction of Section 5, another noncommutative analogue of the Cartier isomorphism is conjugate filtration, which we will address in this section. As there, our version is constructed out of THH(k)-modules for a base commutative ring k, and by Lemma 5.5, it is crucial to analyze the map $\operatorname{THH}(k)^{\Phi C_p} \to k^{tC_p}$, endowing the target k^{tC_p} a suitable filtration. First, we note that, the homotopy C_p -fixed points k^{hC_p} of a commutative \mathbb{F}_p -

algebra k is a direct summand of the C_p -Tate construction k^{tC_p} .

Remark 6.1. Let M be an \mathbb{F}_p -vector space. Then it follows from computations that the composite map

$$M^{hC_p} \longrightarrow M^{tC_p} \longrightarrow \tau_{\leq 0} M^{tC_p}$$

of $\mathbb{F}_n^{hC_p}\text{-}\mathrm{module}$ spectra is an equivalence. In particular, let k be a commutative \mathbb{F}_p -algebra. Then the composite map

$$k^{hC_p} \longrightarrow k^{tC_p} \longrightarrow \tau_{\leq 0} k^{tC_p}$$

of k^{hC_p} -module spectra is an equivalence, where the first map has an \mathbb{E}_{∞} -structure, and the second map has a k^{tC_p} -module structure.

Recall that, in Section 5, a key fact is that the \mathbb{T}/C_p -equivariant \mathbb{Z}^{tC_p} -module spectrum $\mathbb{F}_p^{tC_p}$ is co-induced. It is natural to ask whether the same holds for k^{tC_p} for any commutative \mathbb{F}_p -algebra k. We do not know the answer, but expect it to be false, at least functorially in k (see Remark 6.6). However, note that the \mathbb{E}_{∞} -ring k^{tC_p} is 2-periodic, and we show that a "fundamental region" of k^{tC_p} with respect to its 2-periodicity is \mathbb{T}/C_p -equivariantly k-linearly co-induced.

Notation 6.2. Let n be a positive integer. Let $\operatorname{CoInd}_e^{\mathbb{T}/C_n}$ denote right adjoint $D(\mathbb{Z}) \to D(\mathbb{Z})^{B(\mathbb{T}/C_n)}$ to the (symmetric monoidal) forgetful functor $D(\mathbb{Z})^{B(\mathbb{T}/C_n)} \to D(\mathbb{Z})^{B(\mathbb{T}/C_n)}$ $D(\mathbb{Z}).$

Construction 6.3. Let M be a spectrum. Endow M with trivial T-action, we get a map $\varpi_p^* M \to M^{hC_p}$ in Fun $(B(\mathbb{T}/C_p), \operatorname{Sp})$, where $\varpi_p: \mathbb{T}/C_p \to *$ is the quotient map, which gives rise to a lax symmetric monoidal lax transformation

$$\varpi_p^*(-) \longrightarrow (-)^{hC_p}$$

between lax symmetric monoidal functors $\operatorname{Sp} \to \operatorname{Fun}(B(\mathbb{T}/C_p), \operatorname{Sp})$. On the other hand, there is a non- \mathbb{T}/C_p -equivariant map $M^{hC_p} \to M$ which, by adjunction, gives rise to a map $M^{hC_p} \to \operatorname{CoInd}_e^{\mathbb{T}/C_p} M$ in $\operatorname{Fun}(B(\mathbb{T}/C_p), \operatorname{Sp})$ which is functorial in $M \in Sp$, and this has a lax symmetric monoidal structure. In summary, we have a composite lax symmetric natural transformation

$$\varpi_p^*(-) \longrightarrow (-)^{hC_p} \longrightarrow \operatorname{CoInd}_e^{\mathbb{T}/C_p}(-)$$

of lax symmetric monoidal functors $\operatorname{Sp} \to \operatorname{Fun}(B(\mathbb{T}/C_p), \operatorname{Sp})$.

Lemma 6.4. Let M be an \mathbb{F}_p -vector space. Then the composite map

$$\tau_{\geqslant -1} M^{hC_p} \longrightarrow M^{hC_p} \longrightarrow \operatorname{CoInd}_e^{\mathbb{T}/C_p} M$$

of \mathbb{T}/C_p -equivariant \mathbb{F}_p -module^{6.1} spectra is an equivalence, where the second map is as in Construction 6.3.

^{6.1.} The \mathbb{F}_p -module structure comes from lax symmetric monoidal structures in Construction 6.3.

Proof. Since the maps in question are already constructed, to check that it is an equivalence, we could pick a free \mathbb{Z} -lift \tilde{M} of M, namely, a free abelian group $\tilde{M} = \mathbb{Z}^{\oplus I}$ with $\tilde{M}/p \cong M$. Then taking *I*-direct sum^{6.2} of the equivalence in Corollary 5.3 and truncating at [-1,0], we get an equivalence as an inverse to the composite map in question.

It follows from lax symmetric monoidal structures in Construction 6.3 and Lemma 6.4 that

Corollary 6.5. Let k be a commutative \mathbb{F}_p -algebra. Then there is a canonical equivalence

$$\tau_{\geq -1} k^{hC_p} \longrightarrow \operatorname{CoInd}_e^{\mathbb{T}/C_p} k$$

of \mathbb{T}/C_p -equivariant k-modules.

Remark 6.6. Let M be an \mathbb{F}_p -vector space. It is unclear whether we can functorially identify M^{tC_p} with the co-induced \mathbb{T}/C_p -equivariant \mathbb{Z} -module spectrum $\operatorname{CoInd}_e^{\mathbb{T}/C_p}(M^{t\mathbb{T}})$ (although we can do it functorially in free \mathbb{Z} -lifts \tilde{M}), let alone \mathbb{F}_p -linearly. The multiplicative structure is even more complicated.

Now we describe the filtration on k^{hC_p} and k^{tC_p} as promised.

Construction 6.7. Let k be a commutative \mathbb{F}_p -algebra. Then we consider the *odd* filtration on \mathbb{T}/C_p -equivariant $\tau_{\geq 0}(k^{tC_p})$ -module spectra k^{hC_p} , k^{tC_p} , and the canonical \mathbb{T}/C_p -equivariant $\tau_{\geq 0}(k^{tC_p})$ -module map $k^{hC_p} \rightarrow k^{tC_p}$, given by the odd parts of the Whitehead filtrations, i.e. for $i \in \mathbb{Z}$, we have

$$\operatorname{Fil}_{\operatorname{odd}}^{i} k^{hC_{p}} := \tau_{\geq 2i-1} k^{hC_{p}};$$

$$\operatorname{Fil}_{\operatorname{odd}}^{i} k^{tC_{p}} := \tau_{\geq 2i-1} k^{tC_{p}}.$$

We note that $\operatorname{Fil}_{\operatorname{odd}}^{>0} k^{hC_p} = 0$, and the maps $\operatorname{gr}_{\operatorname{odd}}^i k^{hC_p} \to \operatorname{gr}_{\operatorname{odd}}^i k^{tC_p}$ are equivalences for $i \leq 0$.

Remark 6.8. Note that $\operatorname{gr}_{\operatorname{odd}}^{0} k^{tC_{p}} = \tau_{\geq -1} k^{hC_{p}}$ is concentrated in degree [-1, 0], by [Lur17, Prop 2.2.1.8], the $\tau_{\geq 0}(k^{tC_{p}})$ -module structure on $\operatorname{gr}_{\operatorname{odd}}^{0} k^{tC_{p}}$ descends canonically to a $\tau_{[0,1]}(k^{tC_{p}}) = k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_{p}$ -module structure, where the animated ring $k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_{p}$ is equipped with trivial \mathbb{T}/C_{p} -action.

We now produce the conjugate filtration for THH(k)-modules. Before this, we need a base-independence result, which says that the de-completed Tate construction over k^{tC_p} coincides with that over $\tau_{\geq 0}(k^{tC_p})$.

Lemma 6.9. Let k be a commutative \mathbb{F}_p -algebra. Then the natural transformation

$$(-)^{\eta_{\tau \ge 0}(k^{tC_p})(\mathbb{T}/C_p)} \longrightarrow (-)^{\eta_{k^{tC_p}}(\mathbb{T}/C_p)}$$

of functors $\operatorname{Mod}_{k^{tC_p}}(\operatorname{Sp}^{g_p} \mathbb{T}) \to D(k)$, induced by the map $\tau_{\geq 0}(k^{tC_p}) \to k^{tC_p}$ in $\operatorname{CAlg}(D(k)^{B(\mathbb{T}/C_p)})$, is an equivalence.

Proof. It suffices to show that, for every compact generator $M = k^{tC_p} \otimes [\mathbb{T}/C_{p^r}] \in Mod_{k^{tC_p}}(\mathrm{Sp}^{g_p \mathbb{T}})^{\aleph_0}$, the assembly map

$$M^{\eta_{\tau \ge 0}(k^{tC_p})(\mathbb{T}/C_p)} \longrightarrow M^{h(\mathbb{T}/C_p)}$$

^{6.2.} Tate construction preserves infinite direct sum of uniformly t-bounded objects.

is an equivalence. Recall that the Whitehead tower of $k^{tC_p} = (\tau_{\geq 0} k^{tC_p})[\sigma^{-1}]$ is a sequential colimit of shifts of $\tau_{\geq 0}(k^{tC_p})$, where $\sigma \in \pi_2(k^{tC_p})$ is a generator. Then the result follows from the fact that $(-)^{h(\mathbb{T}/C_p)}$ preserves weakly Whitehead towers (cf. [NS18, Lem I.2.6], or the dual of [BMS19, Lem 3.3]).

Construction 6.10. (Conjugate filtration) Let k be a commutative \mathbb{F}_{p} -algebra, and M a THH(k)-module in Sp^{$g \leq T$}. Then by Lemma 5.5 and Construction 6.7, the $\mathbb{Z}^{t\mathbb{T}}$ -module spectrum

$$(M \otimes_{\mathrm{THH}(k)}^{\mathbb{L}} \underline{k})^{\theta_{\underline{k}} \mathbb{T}} \simeq (M^{\Phi C_p} \otimes_{\mathrm{THH}(k)}^{\mathbb{L}} \phi_{C_p} \tau_{\geq 0}(k^{tC_p}) \otimes_{\tau_{\geq 0}(k^{tC_p})}^{\mathbb{L}} k^{tC_p})^{\eta_{\tau_{\geq 0}(k^{tC_p})}(\mathbb{T}/C_p)}$$

admits a filtration $\operatorname{Fil}_{\operatorname{conj}}^*$ induced by the odd filtration on k^{tC_p} .

We now analyze the associated graded pieces of the conjugate filtration. It suffices to analyze the zeroth associated piece:

Remark 6.11. By 2-periodicity of k^{tC_p} , all $\operatorname{gr}_{\operatorname{odd}}^i k^{tC_p}$'s, as $k^{\Phi C_p}$ -modules, are isomorphic up to a shift. Thus to study associated graded pieces of odd filtered k^{tC_p} , it suffices to study $\operatorname{gr}_{\operatorname{odd}}^0 k^{tC_p} \simeq \operatorname{gr}_{\operatorname{odd}}^i k^{hC_p}$, thus the same for the conjugate filtration.

Since $k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p = (\tau_{\geq 0} k^{tC_p}) / \sigma$ is a perfect $\tau_{\geq 0} k^{tC_p}$ -module in $\operatorname{Sp}^{B(\mathbb{T}/C_p)}$, we have

Lemma 6.12. Let k be a commutative \mathbb{F}_p -algebra. Then the natural transformation

$$(-)^{\eta_{\tau \ge 0}(k^{tC_p})(\mathbb{T}/C_p)} \longrightarrow (-)^{\eta_{k \otimes \mathbb{Z}\mathbb{F}_p}(\mathbb{T}/C_p)}$$

of functors $\operatorname{Mod}_{k\otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{F}_p}(\operatorname{Sp}^{g_p\mathbb{T}}) \to D(k)$, induced by the map $\tau_{\geq 0}(k^{tC_p}) \to k\otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{F}_p$ in $\operatorname{CAlg}(D(k)^{B(\mathbb{T}/C_p)})$, is an equivalence (we tacitly used Remark 2.4).

Similarly, by perfectness of k-module $k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p$, we have

Lemma 6.13. Let k be a commutative \mathbb{F}_p -algebra. Then the natural transformation

$$(-)^{\eta_k(\mathbb{T}/C_p)} \longrightarrow (-)^{\eta_{k\otimes\mathbb{Z}\mathbb{F}_p}(\mathbb{T}/C_p)}$$

of functors $\operatorname{Mod}_{k\otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{F}_p}(\operatorname{Sp}^{g_p\mathbb{T}}) \to D(k)$, induced by the map $k \to k \otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{F}_p$ in $\operatorname{CAlg}(D(k)^{B(\mathbb{T}/C_p)})$, is an equivalence (we tacitly used Remark 2.4).

It follows from Remark 6.8 and Lemmas 6.12 and 6.13 that

Lemma 6.14. Let k be a commutative \mathbb{F}_p -algebra, and M a THH(k)-module in $\operatorname{Sp}^{g^{\leq}\mathbb{T}}$. Then the 0-th associated graded piece $\operatorname{gr}^0_{\operatorname{conj}}(M \otimes_{\operatorname{THH}(k)}^{\mathbb{L}} \underline{k})^{\theta_{\underline{k}}} {}^{\mathbb{T}}$ of the conjugate filtration is equivalent to

$$\left(\left(M^{\Phi C_p} \otimes_{\mathrm{THH}(k)}^{\mathbb{L}} \Phi^{C_p}\left(k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p\right)\right) \otimes_{k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p}^{\mathbb{L}} \mathrm{gr}_{\mathrm{odd}}^{0} k^{t C_p}\right)^{\eta_k(\mathbb{T}/C_p)}$$

in D(k) (again, we tacitly used Remark 2.4).

Now we embark to understand the composite map $\operatorname{THH}(k) \xrightarrow{\simeq} \operatorname{THH}(k)^{\Phi C_p} \to \underline{k} \otimes_{\mathbb{Z}}^{\mathbb{Z}} \mathbb{F}_p$ of $\mathbb{T}\text{-}\mathbb{E}_{\infty}\text{-rings}^{6.3}$. In fact, this could be understood for any commutative ring, not necessarily over \mathbb{F}_p .

^{6.3.} For our purposes, thanks to Remark 2.4, it suffices to understand the underlying T-equivariant \mathbb{E}_{∞} -ring. However, here we deduce slightly stronger results.

Remark 6.15. Let k be an animated ring. By construction (cf. [NS18, §IV.2]), we have a commutative diagram

where the leftmost row are maps of \mathbb{E}_{∞} -rings while the rest are \mathbb{T} - \mathbb{E}_{∞} -rings. The composite map $k \to k^{tC_p}$ from the top left to the bottom right is the Tate-valued Frobenius.

Remark 6.16. Let k be an animated ring. Note that we have a commutative diagram

$$\begin{array}{cccc} k & \longrightarrow & \underline{k}^{\Phi C_p} \\ \downarrow & & \downarrow \\ \underline{k} & \stackrel{\underline{\tilde{\varphi}_k}}{\longrightarrow} & \underline{k} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p \end{array}$$

in $\operatorname{CAlg}(D(\mathbb{Z}))$, where the top arrow is found in Remark 6.15, the bottom arrow is induced by the "extended" Frobenius map $\tilde{\varphi}_k : k \to k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p \to k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p$ of animated rings, and except the top left k, other terms are $\mathbb{T}\text{-}\mathbb{E}_{\infty}$ -rings, and the arrows between $\mathbb{T}\text{-}\mathbb{E}_{\infty}$ -rings have $\mathbb{T}\text{-}\mathbb{E}_{\infty}$ -structures. By the universal property of THH in [ABG+18], we get a commutative diagram

$$\begin{array}{ccc} \operatorname{THH}(k) & \longrightarrow & \underline{k}^{\Phi C_p} \\ \downarrow & & \downarrow \\ & \underline{k} & \xrightarrow{\underline{\tilde{\varphi}}_k} & \underline{k \otimes \mathbb{Z} \, \mathbb{F}_p} \end{array}$$

of \mathbb{T} - \mathbb{E}_{∞} -rings^{6.4}.

So far, we have analyze the part $M^{\Phi C_p} \otimes_{\mathrm{THH}(k)}^{\mathbb{L}} \Phi^{C_p}(k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p)$ of the expression in Lemma 6.14. Now we analyze the part $((-) \otimes_{k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p}^{\mathbb{L}} \operatorname{gr}_{\mathrm{odd}}^0 k^{tC_p})^{\eta_k(\mathbb{T}/C_p)}$ for commutative \mathbb{F}_p -algebras k. It follows from Corollary 6.5 that this simplifies to $(-) \otimes_{k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p}^{\mathbb{L}} k$, where the $k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p$ -module structure on k is simply given by the multiplication map $k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p \to k$. Recall that, for animated \mathbb{F}_p -algebras k, the composite map

$$k \xrightarrow{\varphi_k} k \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p \longrightarrow k$$

coincides with the usual Frobenius map $\varphi_k: k \to k$. The above discussion implies that

Proposition 6.17. Let k be a commutative \mathbb{F}_p -algebra, and M a THH(k)-module in $\operatorname{Sp}^{g^{\leq}\mathbb{T}}$. Then the 0-th associated graded piece $\operatorname{gr}_{\operatorname{conj}}^0(M \otimes_{\operatorname{THH}(k)}^{\mathbb{L}} \underline{k})^{\theta_{\underline{k}}} \mathbb{T}$ of the conjugate filtration is equivalent to the Frobenius twist

$$(M^{\Phi C_p} \otimes_{\mathrm{THH}(k)}^{\mathbb{L}} k) \otimes_{k,\varphi_k}^{\mathbb{L}} k$$

in D(k), where the THH(k)-module structure on $M^{\Phi C_p}$ is induced by the equivalence THH(k) $\xrightarrow{\simeq}$ THH(k) $\xrightarrow{\Phi C_p}$.

^{6.4.} Those who are not familiar with \mathbb{T} - \mathbb{E}_{∞} -rings can simply replace them by \mathbb{T} -equivariant \mathbb{E}_{∞} -rings, and the THH also has an analogous universal property by McClure–Schwänzl–Vogt, cf. [NS18, Prop IV.2.2].

Corollary 6.18. Let k be a commutative \mathbb{F}_p -algebra, and C a dualizable presentable stable \mathbb{F}_p -linear ∞ -category. Then the 0-th associated graded piece $\operatorname{gr}_{\operatorname{conj}}^0\operatorname{HP}^{\operatorname{poly}}(\mathcal{C}/k)$ is equivalent to $\operatorname{HH}(\mathcal{C}/k) \otimes_{k, \varphi_k}^{\mathbb{L}} k$ in D(k).

Finally, we show that the conjugate filtration is complete. It suffices to establish the following connectivity result.

Lemma 6.19. Let k be a commutative \mathbb{F}_p -algebra, and M a $\text{THH}(k)^{\Phi C_p}$ -module in $\text{Sp}^{g^{\leq}(\mathbb{T}/C_p)}$. Suppose that M is connective. Then the spectrum

$$\left(M \otimes_{\mathrm{THH}(k)}^{\mathbb{L}} \tau_{\geq -1} k^{tC_p}\right)^{\eta_{\tau_{\geq 0}\left(k^{tC_p}\right)}\left(\mathbb{T}/C_p\right)}$$

 $is \ connective.$

Note that k^{tC_p} is co-induced in $D(\mathbb{Z})$. Although it might not be compatibility with $\text{THH}(k)^{\Phi C_p}$ -module structure, this is already enough for us to prove the connectivity.

Proof. By the bar resolution of the relative tensor product, and the fact that the de-completed Tate construction preserves small colimits, it suffices to show that, for every $n \in \mathbb{N}$, the spectrum

$$\left(M \otimes_{\underline{\mathbb{Z}}}^{\mathbb{L}} (\mathrm{THH}(k)^{\Phi C_p})^{\otimes_{\underline{\mathbb{Z}}}^{\mathbb{L}} n} \otimes_{\underline{\mathbb{Z}}}^{\mathbb{L}} \tau_{\geq -1} k^{t C_p}\right)^{\eta_{\tau_{\geq 0}\left(k^{t C_p}\right)}(\mathbb{T}/C_p)}$$

is connective. We write k as a direct sum $\mathbb{F}_p^{\oplus I}$, and applying Corollary 5.3, we see that this spectrum is equivalent to the direct sum of I's copies of

$$M \otimes_{\mathbb{Z}}^{\mathbb{L}} (\mathrm{THH}(k)^{\Phi C_p}) \otimes_{\mathbb{Z}}^{\mathbb{L}} n \otimes_{\mathbb{Z}}^{\mathbb{L}} \tau_{\geq 0} \mathbb{Z}^{tC_p}$$

 \square

which is connective.

By 2-periodicity of $k^{tC_p}\!,$ and the fact that $(-)^{\Phi C_p}$ preserves connectivity, we deduce from Lemma 6.19 that

Corollary 6.20. Let k be a commutative \mathbb{F}_p -algebra, and M a connective $\mathrm{THH}(k)$ module in $\mathrm{Sp}^{g^{\leq}\mathbb{T}}$. Then the conjugate filtration on $(M \otimes_{\mathrm{THH}(k)}^{\mathbb{L}} k)^{\eta_k} \mathbb{T}$ is complete.

Corollary 6.21. Let k be a commutative \mathbb{F}_p -algebra, and C a dualizable presentable stable \mathbb{F}_p -linear ∞ -category. Suppose that its topological Hocschild homology THH(C) is bounded below^{6.5}. Then the conjugate filtration on its polynomial periodic cyclic homology HP^{poly}(C/k) is complete.

APPENDIX A. TATE COHOMOLOGY COMPLEX

We briefly show that our de-completed Tate cohomology on Lazard-semi-flat chain complexes of cohomological Mackey functors can be computed by *Tate cohomology complex* in [Kal15, §6.2], or [PVV18, §2.1], thus it is a homotopy invariant version of the latter.

Let k be a commutative ring. Recall that a finitely generated permutation Gmodule is a (left) k[G]-module of the form k[X] for some finite G-set X. We denote by $\operatorname{Perm}_G(k) \subseteq \operatorname{LMod}_{k[G]}$ the full subcategory spanned by finitely generated permutation G-modules. We refer to [BG21] for comparison of different characterizations of cohomological Mackey functors, and [BCN21, Ex 2.5] for the equivalence $D(\operatorname{Mack}_{G}^{\operatorname{coh}}(k)) = \operatorname{Mod}_k(\operatorname{Sp}^{gG})$ between derived cohomological G-Mackey functors and <u>k</u>-modules in genuine G-spectra.

^{6.5.} This is the case when C = D(R) for a (-1)-connective \mathbb{E}_1 -k-algebra R, or C = D(X) for a quasicompact quasiseparated k-scheme X.

Now we start with a flatness of cohomological Mackey functors.

Definition A.1. Let k be a commutative ring, and G a finite group.

- We say that a cohomological Mackey functor is Lazard-flat if it is a filtered colimit of finitely generated permutation modules.
- We say that an (unbounded) chain complex of cohomological Mackey functors is Lazard-semi-flat if it is a filtered colimit of bounded chain complexes of finitely generated permutation modules. Compare with [CH15, Thm 1.1].

Remark A.2. Let k be a commutative ring, and G a finite group. Then permutation k[G]-modules are compact in the category $\operatorname{Mack}_{G}^{\operatorname{coh}}(k)$ of cohomological G-Mackey functors. Consequently, we have a canonical fully faithful functor

$$\operatorname{Ind}(\operatorname{Perm}_G) \hookrightarrow \operatorname{Mack}_G^{\operatorname{coh}}(k).$$

In particular, we can identify Lazard-flat (derived) cohomological Mackey functors as objects of $\operatorname{Ind}(\operatorname{Perm}_G(k))$.

Similarly, recall that a chain complex in an additive category is compact if and only if it is bounded and degreewise compact [CH15, Thm 4.5]. The preceding argument shows that we have a canonical fully faithful functor

$$\operatorname{Ind}(\operatorname{Ch}^{b}(\operatorname{Perm}_{G}(k))) \hookrightarrow \operatorname{Ch}(\operatorname{Mack}_{G}^{\operatorname{coh}}(k))$$

and thus we can identify Lazard-semi-flat chain complexes of cohomological Mackey functors as objects of $\operatorname{Ind}(\operatorname{Ch}^{b}(\operatorname{Perm}_{G}(k)))$.

Lazard-flat cohomological Mackey functors (resp. Lazard-semi-flat chain complexes of cohomological Mackey functors) are in fact k[G]-modules (resp. chain complexes of k[G]-modules):

Remark A.3. Let k be a commutative ring, and G a finite group. Then finitely generated permutation k[G]-modules are compact in the category $\operatorname{LMod}_{k[G]}$ of left G-modules, thus the fully faithful functor

$$\operatorname{Ind}(\operatorname{Perm}_G(k)) \hookrightarrow \operatorname{Mack}_G^{\operatorname{coh}}(k)$$

in Remark A.2 factors through the inclusion $\operatorname{LMod}_{k[G]} \hookrightarrow \operatorname{Mack}_G^{\operatorname{coh}}(k),$ and the fully faithful functor

$$\operatorname{Ind}(\operatorname{Ch}^{b}(\operatorname{Perm}_{G}(k))) \hookrightarrow \operatorname{Ch}(\operatorname{Mack}_{G}^{\operatorname{coh}}(k))$$

factors through the inclusion $\operatorname{Ch}(\operatorname{LMod}_{k[G]}) \hookrightarrow \operatorname{Ch}(\operatorname{Mack}_{G}^{\operatorname{coh}}(k)).$

Now we review the Tate cohomology complex. Let k be a commutative ring, and G a finite group. A complete resolution [PVV18, §2.1] of the left k[G]-module k is an acyclic complex $P_* \in Ch(LMod_{k[G]}^{free})$ of free left k[G]-modules along with an isomorphism $\varepsilon : \mathbb{Z} \to ker(d: P_0 \to P_{-1})$ of left k[G]-modules.

Definition A.4. Let k be a commutative ring, G a finite group, and P_* a complete resolution of k. The functor of the Tate cohomology complex is defined to be the functor

$$\begin{array}{rcl} \operatorname{Ch}(\operatorname{Mack}_{G}^{\operatorname{coh}}(k)) &\longrightarrow & \operatorname{Ch}(k), \\ & M_{*} &\longmapsto & (\operatorname{Tot}^{\oplus}(M_{*} \otimes_{k} P_{*}))^{G}. \end{array}$$

On Lazard-semi-flat chain complexes of cohomological Mackey functors, the Tate cohomology complex represents the de-completed Tate cohomology. More precisely, we have

Proposition A.5. Let k be a commutative ring, and G a finite group. Then we have a commutative diagram

$$\begin{array}{cccc}
\operatorname{Ind}(\operatorname{Ch}^{b}(\operatorname{Perm}_{G}(k))) & \longrightarrow & \operatorname{Ch}(k) \\
\downarrow & & \downarrow \\
D(\operatorname{Mack}_{G}^{\operatorname{coh}}(k)) = \operatorname{Mod}_{\underline{k}}(\operatorname{Sp}^{gG}) & \xrightarrow{(-)^{\theta_{\underline{k}}G}} & D(k)
\end{array}$$

of additive ∞ -categories, where the top horizontal arrow is the composite of the inclusion in Remark A.2 and the Tate cohomology complex, and the vertical arrows are localizations at quasi-isomorphisms.

Sketch of proof. Since all functors in question preserve filtered colimits, we can replace the top left term $\operatorname{Ind}(\operatorname{Ch}^b(\operatorname{Perm}_G(k)))$ by $\operatorname{Ch}^b(\operatorname{Perm}_G(k))$. By stability, we could further restrict to the full subcategory $\operatorname{Ch}_{\geq 0}^b(\operatorname{Perm}_G(k))$ of bounded chain complexes concentrated in non-negative degrees. This is the same as freely adjoining finitary-geometric-realizations to $\operatorname{Perm}_G(k)$. All functors in question preserve finitary geometric realizations, thus we can replace the top left term simply by $\operatorname{Perm}_G(k)$.

In this case, the left vertical arrow is fully faithful, with essential image being compact in $\operatorname{Mod}_{\underline{k}}(\operatorname{Sp}^{gG})$. Therefore we may replace $(-)^{\theta_{\underline{k}}G}$ by $(-)^{tG}$. It remains to show that, for every finitely generated permutation *G*-module *M*, the complex $(M \otimes_{\underline{k}} P_*)^G$ represents M^{tG} , which follows from definition.

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