Revisiting derived crystalline cohomology

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1 Surjections of animated rings

Apps: classical objs w/ *acyclicity conds* w/o finiteness:

Definition. (Quillen) (A, I) *is* quasiregular (abbrev. quasireg.) $\overset{\triangle}{\iff}$ $\mathbb{L}_{(A/I)/A}[-1]$ *is* A/I*-flat.*

Theorem 1. (ans. to Illusie's Q; M.)

- (A, I) — *a quasireg. pair;*
- $(B, J, \gamma) :=$ *the PD-env. of* (A, I).

$\Longrightarrow$ *the canon. map* $\Gamma^*_{A/I}(I/I^2) \to J^{[*]}/J^{[*+1]}$ *is an equiv.*

Definition. *A map* $R \to S$ *in* Ani(Ring) *is* surj. $\overset{\triangle}{\iff}$ $\pi_0(R) \to \pi_0(S)$ *is surj.*

Notation. $\mathrm{An} := \{\textit{Kan complexes}\}[(\textit{homotopy equivs})^{-1}]$.

Definition. (Quillen, Lurie) $\mathcal{C}$ — *a small* ∞*-cat. w/ fin. coprods. The* nonab. deriv. cat. $\mathcal{P}_\Sigma(\mathcal{C}) := \{F : \mathcal{C}^{\mathrm{op}} \to \mathrm{An} \mid F(\varnothing) \simeq \{*\}$ *and* $F(X \amalg Y) \simeq F(X) \times F(Y)\}$[1].

Proposition. (Quillen, Lurie)

- $\mathcal{C}$ — *a small* ∞*-cat. w/ fin. coprods;*
- $\mathcal{D}$ — *a cocpl.* ∞*-cat.*

$\Longrightarrow$ $\exists$ *an equiv.*

$$\{G : \mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{D} \mid G \text{ preserves fil. colims. \& geom. reals}\} \underset{\mathbb{L} := \mathrm{Lan}_{\mathcal{C} \hookrightarrow \mathcal{P}_\Sigma(\mathcal{C})}}{\overset{\mathrm{res}}{\rightleftarrows}} \{F : \mathcal{C} \to \mathcal{D}\}$$

of ∞*-cats.* $\mathbb{L}F$ — *the* left deriv. fun. *of* F.

Definition. (Lurie) *An* ∞*-cat.* $\mathcal{D}$ *is* proj. gen. $\overset{\triangle}{\iff}$ $\exists\, \mathcal{C} \subseteq_{\mathrm{full}} \mathcal{D} : \mathcal{P}_\Sigma(\mathcal{C}) \xrightarrow{\simeq} \mathcal{D}$.

1. In the talk, the condition $F(\varnothing) = \{*\}$ was mistakenly omitted. Thanks for Ofer GABBER for pointing out this.

Example. Proj. gen.

1. $\mathrm{Ani}(\mathrm{Ring}) = \mathcal{P}_\Sigma(\mathrm{Poly})$;

2. $\forall Z \in \mathcal{P}_\Sigma(\mathcal{C}) : \mathcal{P}_\Sigma(\mathcal{C})_{Z/} \simeq \mathcal{P}_\Sigma(\{Z \to Z \amalg X \mid X \in \mathcal{C}\})$.

Theorem 2. (M.) $\mathrm{AniPair} := \{A \twoheadrightarrow A'' \mid A, A'' \in \mathrm{Ani}(\mathrm{Ring})\} \simeq \mathcal{P}_\Sigma(\mathrm{StPair})$ *where* $\mathrm{StPair} := \{\mathbb{Z}[X, Y] \twoheadrightarrow \mathbb{Z}[X] \mid X, Y \in \mathrm{Fin}\}$.

Definition.

1. $\mathrm{AniPDPair} := \mathcal{P}_\Sigma(\mathrm{StPDPair})$ *where* $\mathrm{StPDPair} := \{(\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X], \gamma) \mid X, Y \in \mathrm{Fin}\}$.
2. $\mathrm{AniPDEnv} := \mathbb{L}(\mathrm{StPair} \to \mathrm{AniPDPair}, (A, I) \mapsto D_A(I))$.

3. *Forgetful:* $\mathbb{L}(\mathrm{StPDPair} \to \mathrm{AniPair}, (A, I, \gamma) \mapsto (A, I))$.

Facts.

1. $\mathrm{Pair} \subseteq_{\mathrm{full}} \mathrm{AniPair}$, $\mathrm{PDPair} \subseteq_{\mathrm{full}} \mathrm{AniPDPair}$.
2. $$\begin{array}{c}\mathrm{AniPair}\\ | \\ (A \twoheadrightarrow A'') \in \mathrm{Pair} \iff \pi_i(A) = \pi_i(A'') = 0 \quad i \neq 0\end{array}$$
3. $$\begin{array}{c}\mathrm{AniPDPair}\\ | \\ (A \twoheadrightarrow A'', \gamma) \in \mathrm{PDPair} \iff (A \twoheadrightarrow A'') \in \mathrm{Pair}\end{array}$$

Outline of pf. of $\Gamma^*_{A/I}(I/I^2) \xrightarrow{\sim} J^{[*]}/J^{[*+1]}$.

1. Animated version:

   a. Ani. Rees alg. $:= \mathbb{L}(\mathrm{StPair} \to \mathrm{Gr}(\mathrm{Ani}(\mathrm{Ring})), (A, I) \mapsto (I^n)_{n \in \mathbb{N}})$.

   b. PD-variant: $\mathbb{L}(\mathrm{StPDPair} \to \mathrm{Gr}(\mathrm{Ani}(\mathrm{Ring})), (A, I, \gamma) \mapsto (I^{[n]})_{n \in \mathbb{N}})$.

   c. Assoc. gr.: $\mathbb{L}(\mathrm{StPDPair} \to \mathrm{Gr}(\mathrm{Ani}(\mathrm{Ring})), (A, I, \gamma) \mapsto (I^{[n]}/I^{[n+1]})_{n \in \mathbb{N}})$.

   $\Longleftarrow$ thm on StPair.

2. $\mathbb{L}$ "$J^{[n]}/J^{[n+1]}$" are static by acyclicity $\Longrightarrow$ so are "$B/J^{[n]}$".

3. Initiality of $B \to B/J^{[*]}$.

$\square$

Theorem 3. (M.) *(A, I) — quasireg. $\mathbb{F}_p$-pair s.t. $(A/I) \otimes^{\mathbb{L}}_{A,\varphi} A$ is static.* $\Longrightarrow$ $\mathrm{AniPDEnv}(A, I) \simeq D_A(I)$.

Notation. $\mathrm{char}\, p$: *$\mathbb{F}_p$ in place of $\mathbb{Z}$. e.g.* $\mathrm{StPair}_{\mathbb{F}_p} := \{\mathbb{F}_p[X, Y] \twoheadrightarrow \mathbb{F}_p[X] \mid X, Y \in \mathrm{Fin}\}$, $\mathrm{StPDPair}_{\mathbb{F}_p}$, $\mathrm{AniPDPair}_{\mathbb{F}_p} := \mathcal{P}_\Sigma(\mathrm{StPDPair}_{\mathbb{F}_p})$.

Remark. $(A, I) \in \mathrm{StPair}_{\mathbb{F}_p} \rightsquigarrow (A/I) \otimes^{\mathbb{L}}_{A,\varphi} A \circlearrowleft D_A(I)$.

Definition. $(A, I) \in \mathrm{StPair}_{\mathbb{F}_p}$. *The* conj. fil. *on $D_A(I)$:*

$$\mathrm{Fil}^{-n}_{\mathrm{conj}}(D_A(I)) := \sum_{\substack{i_1 + \cdots + i_m \leq n \\ f_1, \ldots, f_m \in I}} ((A/I) \otimes^{\mathbb{L}}_{A,\varphi} A)\, \gamma_{i_1 p}(f_1) \cdots \gamma_{i_m p}(f_m) \subseteq D_A(I)$$

$\rightsquigarrow$ *a functor* $\mathrm{StPair}_{\mathbb{F}_p} \to \mathrm{Fil}^{\leq 0}(\mathrm{Ani}(\mathrm{Ring})) \rightsquigarrow \mathbb{L}(\cdots) : \mathrm{AniPair}_{\mathbb{F}_p} \to \mathrm{Fil}^{\leq 0}(\mathrm{Ani}(\mathrm{Ring}))$, *also the* conj. fil. $\mathrm{Fil}_{\mathrm{conj}}$.

Lemma. (Bhatt) $\forall (A, I) \in \mathrm{StPair}_{\mathbb{F}_p}$, $\exists$ *a comp. isom.* $\Gamma^*_{A/I}(I/I^2) \otimes^{\mathbb{L}}_{A,\varphi} A \to \mathrm{gr}^{-*}_{\mathrm{conj}}(D_A(I))$ *of $(A/I) \otimes^{\mathbb{L}}_{A,\varphi} A$-mods, functorial in (A, I).*

Outline of proof of $\mathrm{AniPDEnv}(A, I) \simeq D_A(I)$.

1. $\mathbb{L}$(the lemma above) + quasireg. $\Longrightarrow$ the canon. $(A/I) \otimes^{\mathbb{L}}_{A,\varphi} A \to \mathrm{AniPDEnv}(A, I)$ is faithfully flat.
2. $(A/I) \otimes^{\mathbb{L}}_{A,\varphi} A$ static $\Longrightarrow$ so is $\mathrm{AniPDEnv}(A, I) \Longrightarrow \in \mathrm{PDPair} \xRightarrow{\text{initiality}}$ result.

□

2 Derived crystalline cohomology and prisms

Theorem. (Bhatt) *$A \to B$ — a l.c.i. map of flat $\mathbb{Z}/p^n$-algs.* $\Longrightarrow$ *the comp. map*

$$\mathrm{dR}_{B/A} \longrightarrow R\Gamma((B/A)_{\mathrm{cris}}, \mathcal{O}_{\mathrm{cris}})$$

is an equiv.

Definition. (BMS2) $A \to R$ quasisyntomic *(abbrev. quasisyn.)* $\overset{\triangle}{\Longleftrightarrow}$ *flat + $\mathbb{L}_{R/A}$ has* Tor-*ampl. in* $[0, 1]$.

Theorem 4. (M.)

- *(A, I, γ) — a PD-pair s.t. $p \in A$ is nilp.*
- *R — a quasisyn. A/I-alg.*

$\Longrightarrow$ *derived crys. coh. of $R/(A, I, \gamma) \simeq$ crys. coh. of $R/(A, I, \gamma)$.*

Lemma. $\mathcal{C}$ *— a small ∞-cat. w/ fin. coprods* $\Longrightarrow \operatorname{Fun}(\Delta^1, \mathcal{P}_\Sigma(\mathcal{C})) \simeq \mathcal{P}_\Sigma(\{X \to X \amalg Y \mid X, Y \in \mathcal{C}\})$.

Corollary. $\mathrm{dRCon} := \operatorname{Fun}(\Delta^1, \mathrm{AniPDPair}) \simeq \mathcal{P}_\Sigma(\mathrm{dRCon}^0)$ *where* $\mathrm{dRCon}^0 := \{(\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X], \gamma) \to (\Gamma_{\mathbb{Z}[X,X']}(Y, Y') \twoheadrightarrow \mathbb{Z}[X, X'], \tilde{\gamma}) \mid X, X', Y, Y' \in \mathrm{Fin}\}$.

Definition. *The* deriv. dR cohom. $\mathrm{dR}_{\cdot/\cdot} : \mathrm{dRCon} \to \mathrm{CAlg}_\mathbb{Z} := \mathbb{L}(\mathrm{dR}_{\cdot/\cdot} : \mathrm{dRCon}^0 \to \mathrm{CAlg}_\mathbb{Z})$ *— for*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ A'' & \longrightarrow & B'' \end{array}$$

Definition. $\mathrm{CrysCoh}_{B''/(A \twoheadrightarrow A'', \gamma_A)} := \mathrm{dR}_{(\mathrm{id}_{B''}, 0)/(A \twoheadrightarrow A'', \gamma_A)}$.

Strategy of pf. of derived crys. comp.

1. Derived sites out of animated PD-pairs.

2. Site coh. $\simeq$ derived site coh., via Čech-Alex.

3. Suffices to show: derived site cohom. $\simeq$ derived crys. coh.
   a. When $A/I \to R$ is surj.: reduces to **(Bhatt)** $\mathrm{dR}_{\mathbb{F}_p/\mathbb{F}_p[x]} \simeq \Gamma_{\mathbb{F}_p}(x)$

   b. Conj. fil. $\Longrightarrow$ desc. w.r.t. base $\xRightarrow{\text{Čech-Alex}}$ result.

□

Theorem 5. (M.) (generalizes results in [Morrow-Tsuji], [Chatzistamation] & [Tian])

$$\begin{array}{ccc} A & \longrightarrow & P \\ \downarrow & & \downarrow\!\!\!\downarrow \\ A/d & \longrightarrow & R \end{array}$$

- (A, d) *— a bnd oriented prism.*
- R *— a p-complete, p-completely quaisyn. A/d-alg.*
- P *— a (p, d)-complete δ-A-alg, (p, d)-completely quasismooth over A*
- $P \twoheadrightarrow R$ *— a surj. of A-algs.*

$\Longrightarrow$ *The prism. env. of $P \twoheadrightarrow R$ is a flat cover of the final obj. in $\Delta\!\!\!\!\Delta(R/A)$.*